Formulae for the exponential, the hyperbolic and the trigonometric functions in terms of the logarithmic function

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Abstract  A common definition of the exponential function is as the inverse function of the logarithmic function, which is defined as the definite integral of the rational function $1/t$ over the interval $[1,x]$ with $x > 0$. The hyperbolic functions (hyperbolic sine, cosine, tangent, etc.) are next defined in terms of the exponential function. Here we derive an explicit real formula for the hyperbolic tangent function in terms of the logarithmic function, which is sufficient for the direct derivation of analogous formulae for the exponential function and the other hyperbolic functions. A similar formula for the trigonometric tangent function, which can be directly used for the derivation of analogous formulae for the other trigonometric functions, is also derived. The present results are based on a simple method for the derivation of closed-form formulae for the zeros of sectionally analytic functions.

Keywords  Exponential function · Hyperbolic functions · Trigonometric functions · Logarithmic function · Elementary transcendental functions · Inverse functions · Closed-form formulae · Sectionally analytic functions · Sectionally meromorphic functions · Zeros and poles.

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1. Introduction

We reconsider the classical problem of the introduction of the elementary transcendental functions in real analysis. A common approach to the introduction of these functions, which is described in sufficient detail even in elementary textbooks on calculus (see, e.g., that by Apostol [2]), is at first to define the logarithmic function, \( \log x \) (or \( \ln x \)), as an integral of the algebraic function \( \frac{1}{x} \):

\[
\log x = \int_1^x \frac{dt}{t}, \quad x > 0.
\]  
(1)

Next, the exponential function, \( \exp x \), is defined for all real values of \( x \) as the inverse function of the logarithmic function, that is, implicitly, through one of the formulae [2]

\[
\log(\exp x) = x, \quad -\infty < x < \infty, \text{ or } \exp(\log x) = x, \quad x > 0.
\]  
(2)

Then the hyperbolic functions can be defined in terms of the exponential function (see, e.g., [2]):

\[
\sinh x = \frac{\exp x - \exp(-x)}{2}, \quad \cosh x = \frac{\exp x + \exp(-x)}{2},
\]  
(3)

\[
\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(2x) - 1}{\exp(2x) + 1}.
\]  
(4)

In [2] it is explained in sufficient detail why the logarithmic function should be defined the first (this is simply due to the fact that the explicit formula (1) is available), whereas the exponential function should be defined as the inverse function of the logarithmic function. (For the exponential function no simple formula, like Eq. (1), valid for every real value of \( x \), is available.)

For the trigonometric functions (sine, cosine, tangent, etc.) it is similarly possible at first to define one inverse trigonometric function, e.g. the inverse sine function, \( \sin^{-1} x \equiv \arcsin x \), through a formula analogous to Eq. (1) (see, e.g., [2]):

\[
\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad -1 \leq x \leq 1.
\]  
(5)

Afterwards, we can define the sine function, \( \sin x \), as the inverse function of the inverse sine function, \( \sin^{-1} x \equiv \arcsin x \), that is, implicitly, though one of the formulae [2]

\[
\sin^{-1}(\sin x) = x, \quad -\pi/2 \leq x \leq \pi/2, \text{ or } \sin(\sin^{-1} x) = x, \quad -1 \leq x \leq 1.
\]  
(6)

Then the other trigonometric functions can be defined in terms of the sine function.

The aforementioned definition of the exponential function (as the inverse function of the logarithmic function), although mathematically both completely correct and easily leading to the properties of this function, yet it has the disadvantage that it is not accompanied by an explicit formula but only by the implicit formula (2). Here we will derive such an explicit formula for the hyperbolic tangent function, which leads directly to a corresponding formula for the exponential function since

\[
\exp x = \sqrt{1 + \tanh x}, \quad -\infty < x < \infty.
\]  
(7)

Similarly, the inverse hyperbolic tangent function \( \tanh^{-1} x \) is directly related to the logarithmic function \( \log x \) through the formulae

\[
\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}, \quad -1 < x < 1,
\]  
(8)

\[
\tanh^{-1} \frac{1}{x} = \frac{1}{2} \log \frac{x+1}{x-1}, \quad x < -1 \text{ or } x > 1.
\]  
(9)
After the derivation of the formula for the hyperbolic tangent function, we will derive an analogous formula for the trigonometric tangent function, which can be used for the derivation of explicit formulae for the other trigonometric functions as well.

We can add that instead of the tangent functions (hyperbolic and trigonometric), we could also work with the sine or cosine functions without any particular difficulty. The same holds true for the exponential functions too.

Finally, the method used for the derivation of the aforementioned explicit formulae is that described in \[1\] and it is really an elementary method based on a corollary of the Cauchy theorem in complex analysis \[3\] although the derived formulae are real. We do not have in mind competitive methods which could be used successfully here.

### 2. The hyperbolic functions

We will prove the following theorem:

**Theorem 1:** The following formula holds true for the hyperbolic tangent function:

\[
\tanh x = \frac{\Theta_0(x) - \frac{2}{x^2}}{2}, \quad -\infty < x < \infty, \quad x \neq 0,
\]

where

\[
\Theta_j(x) = \int_{-1}^{1} \frac{t^j}{m(t, x)} \, dt, \quad j = 0, 1,
\]

with

\[
m(t, x) = (x - \tanh^{-1} t)^2 + \left(\frac{\pi}{2}\right)^2.
\]

**Proof:** We consider the function \( y = \tanh x \) \((-\infty < x < \infty)\) and its inverse function \( x = \tanh^{-1} y \) \((-1 < y < 1)\) or better \( x = \tanh^{-1}(1/u) \) with \( y = 1/u \) \((u < -1 \text{ or } u > 1)\). We seek the pole \( a \) (in the complex plane) of the sectionally meromorphic function

\[
M(z) = \frac{1}{x - \tanh^{-1} \frac{1}{z}}, \quad z = u + iv,
\]

which has the interval \([-1, 1]\) as a discontinuity interval. Then, clearly, \( y = 1/a \).

Now, following the method described in \[1\] for the location of poles of sectionally meromorphic functions (or zeros of sectionally analytic functions), we consider the complex contour integral

\[
I = \oint_C (z - a)M(z) \, dz,
\]

where the closed sectionally smooth contour \( C \) is assumed surrounding the discontinuity interval \([-1, 1]\) of \( M(z) \). Since, evidently, \((z - a)M(z)\) is an analytic function in the cut complex plane (without poles), the value of \( I \) remains unchanged if we let \( C \) tend to infinity or shrink onto the interval \([-1, 1]\) \[3\].

In the first case, taking into account that

\[
\tanh^{-1} \frac{1}{z} = \frac{1}{z} + O\left(\frac{1}{z^3}\right), \quad z \to \infty,
\]

\[
\Theta_0(x) = \int_{-1}^{1} \frac{1}{1 - \tanh^2 t} \, dt,
\]

and

\[
\Theta_1(x) = 2 \int_{-1}^{1} \frac{t}{1 - \tanh^2 t} \, dt,
\]

we have

\[
\Theta_0(x) = \frac{\pi}{2} \quad \text{and} \quad \Theta_1(x) = \pi.
\]
we find directly that
\[ M(z) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + O\left(\frac{1}{z^3}\right), \quad z \to \infty. \quad (16) \]

On the other hand, for an integer \( k \) it is well known that
\[ \oint_C z^k \, dz = 0, \quad k \neq -1 \quad \text{and} \quad \oint_C z^{-1} \, dz = 2\pi i. \quad (17) \]

Then we find directly from Eq. (14) that
\[ I = 2\pi i \left(\frac{1}{x^3} - \frac{a}{x^2}\right). \quad (18) \]

In the second case, i.e. when \( C \) shrinks onto \([-1, 1]\), taking into account the boundary values of \( \tanh^{-1}(1/z) \) on \([-1, 1]\), i.e.
\[ \left[ \tanh^{-1} \frac{1}{t} \right]^{\pm} = \tanh^{-1} t \mp \frac{\pi i}{2}, \quad -1 < t < 1, \quad (19) \]
for the corresponding values of \( M(z) \) we find directly that
\[ M^{\pm}(t) = \frac{x - \tanh^{-1} t \mp \frac{\pi i}{2}}{m(t,x)}, \quad -1 < t < 1, \quad (20) \]
with \( m(t,x) \) given by Eq. (12). Therefore,
\[ M^{+}(t) - M^{-}(t) = -\frac{\pi i}{m(t,x)}, \quad -1 < t < 1, \quad (21) \]
and, finally,
\[ I = \pi i [\Theta_1(x) - a\Theta_0(x)] \quad (22) \]
with \( \Theta_j(x) \) given by Eqs. (11) and the positive direction on \([-1, 1]\) (from \(-1\) to \(+1\)) having been taken into consideration.

Now, by comparing Eqs. (18) and (22), we find that
\[ a = \frac{\Theta_1(x) - \frac{2}{x^3}}{\Theta_0(x) - \frac{2}{x^2}}, \quad -\infty < x < \infty, \quad x \neq 0, \quad (23) \]
and, since \( y = \tanh x = 1/a \), Eq. (10) follows. \( \square \)

Remark 1: Because of Eq. (8), Eq. (10) can be considered to be a formula for the hyperbolic tangent function in terms of the logarithmic function.

Remark 2: Because of Eqs. (7) and (3), Eq. (10) can be directly used for the derivation of similar formulae for the exponential function and the hyperbolic sine and cosine functions.

Remark 3: Analogous results to those of this section could be obtained if we were working directly with the exponential function or with the hyperbolic sine or cosine functions. The final formulae would be different in appearance (although essentially equivalent) to the corresponding formulae obtained on the basis of Eq. (10) as was already explained in Remark 2.

Now we proceed to a similar consideration of the trigonometric functions.
3. The trigonometric functions

We will prove the following theorem, which is analogous to Theorem 1 in the previous section:

**Theorem 2:** The following formula holds true for the trigonometric tangent function:

\[
\tan x = \frac{J_0(x) - \pi}{J_1(x) - \frac{\pi}{x^2}}, \quad -\pi/2 < x < \pi/2, \quad x \neq 0,
\]  

(24)

where

\[
J_0(x) = \int_0^1 \left[ \frac{x-(\pi/2)}{m^+(t,x)} - \frac{x+(\pi/2)}{m^-(t,x)} \right] dt,
\]

(25)

\[
J_1(x) = \int_0^1 t \tanh^{-1} t \left[ \frac{1}{m^+(t,x)} - \frac{1}{m^-(t,x)} \right] dt
\]

(26)

with

\[
m^\pm(t,x) = [x \mp (\pi/2)]^2 + (\tanh^{-1} t)^2.
\]

(27)

**Proof.** The proof of this theorem is analogous to the proof of Theorem 1. We consider the function \( y = \tan x \) \((-\pi/2 < x < \pi/2)\) and its inverse function \( x = \tan^{-1} y \) \((-\infty < y < \infty)\) or better \( x = itanh^{-1}(1/u) \) with \( y = i/u \) \((-\infty < u < \infty)\). We seek the pole \( a \) (in the complex plane) of the sectionally meromorphic function

\[
M(z) = \frac{1}{x - itanh^{-1} \frac{1}{z}}, \quad z = u + iv.
\]

(28)

Then, clearly, \( y = i/a \).

To this end, we consider again the integral (14) but with \( M(z) \) given now by the above Eq. (28). We take also into account Eq. (15) and we find that

\[
M(z) = \frac{1}{x} + \frac{i}{x^2 z} - \frac{1}{x^3 z^2} + O\left(\frac{1}{z^3}\right), \quad z \to \infty
\]

(29)

instead of Eq. (16). Next, because of Eqs. (17) and (28), we find directly from Eq. (14) that

\[
I = -2\pi i\left(\frac{1}{x^3} + i \frac{a}{x^2}\right).
\]

(30)

On the other hand, by taking into consideration Eqs. (19), we obtain from Eq. (28)

\[
M^\pm(t) = \frac{x \mp (\pi/2) + itanh^{-1} t}{m^\pm(t,x)}, \quad -1 < t < 1,
\]

(31)

with \( m^\pm(t,x) \) given by Eqs. (27). Then by letting the closed contour \( C \) shrink onto the discontinuity interval \([-1, 1]\) of \( M(z) \), it is directly seen from Eq. (14) that

\[
I = -2iJ_1(x) + 2aJ_0(x)
\]

(32)

with \( J_j(x) \) given by Eqs. (25) and (26).

Now, by comparing Eqs. (30) and (32), we find directly that

\[
a = i \frac{J_1(x)}{J_0(x)}, \quad -\pi/2 < x < \pi/2, \quad x \neq 0,
\]

(33)

and, since \( y = \tan x = i/a \), Eq. (24) follows. □
Remark 4: Remarks 1 to 3 of the previous section concerning hyperbolic functions also apply here to the corresponding trigonometric functions as well as to the related complex exponential function exp(ix).

4. Verification of the formulae

The validity of the fundamental formulae (10) and (24) was also tested numerically for several values of \( x \). The integrals \( \Theta_j(x) \) and \( J_j(x) \) (with \( j = 0, 1 \)) were computed by using the classical Gauss quadrature rule [4] with \( n = 2, 4, 8 \) and 16 nodes on the intervals \([-1, 1]\) and \([0, 1]\), respectively. The nodes and weights used are also tabulated in [4]. The obtained numerical results (not displayed here) were seen to be sufficiently accurate for just moderate values of \( n \). Moreover, clearly, the integrands in \( \Theta_j(x) \) and \( J_j(x) \) (with \( j = 0, 1 \)) are well-behaved continuous functions on the corresponding integration intervals. This assures the rapid convergence of the results of the Gaussian quadrature rules used to the corresponding exact values of the integrals [4].

References