Application of *Mathematica* to the Rayleigh–Ritz method for plane elasticity problems

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**Abstract** The classical Rayleigh–Ritz method for plane isotropic elasticity problems governed by the well-known biharmonic equation (satisfied by the Airy stress function) is revisited. The modern and powerful computer algebra system *Mathematica* was employed for the symbolic/numerical approximate solution of the biharmonic equation. A related simple procedure was prepared and the classical problem of a rectangular elastic region loaded by a parabolic tensile loading was chosen as an example of the application of the approach. The available symbolic/numerical results in the literature and additional more complicated analogous results were directly derived by using the aforementioned procedure. Further related possibilities and generalizations are also discussed in brief.

**Keywords** Rayleigh–Ritz method · Plane elasticity · Isotropic elasticity · Biharmonic equation · Airy stress function · Symbolic computations · Numerical computations · Approximate solutions · Computer algebra · *Mathematica*

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**Final publication details** This technical report was presented to the 3rd National Congress on Mechanics, Athens, June 25–27, 1992. This Congress was organized by the Hellenic Society for Theoretical & Applied Mechanics (HSTAM). Therefore, the official publication of the present technical report is that included in the Proceedings of this Congress with the following details:

Although the style of this final, official publication is somewhat different from the style of the present technical report, nevertheless, the contents of the final publication essentially coincide with the contents of the present technical report.

1. Introduction–the approach

The Rayleigh–Ritz method has been extensively used in a variety of problems in mechanics, including plane isotropic elasticity problems, and it reduces such a problem to a system of linear algebraic equations leading to the approximation of the exact solution. Standard related references are the classical books on elasticity by Timoshenko and Goodier [1] and Sokolnikoff [2], where both the theory and applications of the Rayleigh–Ritz method are presented. This method leads to the problem of minimization of the fundamental quantity [1]

\[ V = \iint_D \left[ \left( \frac{\partial^2 \phi}{\partial y^2} \right)^2 + \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] \, dx \, dy \]  

(1)

on the domain \( D \) of the elastic medium in a Cartesian coordinate system \( Oxy \). This quantity, \( V \), is proportional to the strain energy of the elastic medium (Timoshenko and Goodier [1]). The minimization of \( V \) by an appropriate selection of the Airy stress function \( \phi = \phi(x,y) \) appearing in Eq. (1) will lead to the approximate solution of the present elasticity problem (if the boundary conditions are satisfied as well), that is, to the approximate satisfaction of the biharmonic equation

\[ \nabla^4 \phi(x,y) \equiv \Delta^2 \phi(x,y) = 0 \]  

by \( \phi(x,y) \).

During the approximate solution of the above minimization problem we choose an appropriate finite series for \( \phi(x,y) \) (with \( n \) terms) and we have to determine the coefficients \( c_i \) of this series by evaluating all the partial derivatives in Eq. (1), the corresponding integrand, the double integral in Eq. (1) (that is, the quantity \( V \) to be minimized) and the partial derivatives of \( V \) with respect to each one of \( c_i \). Next, we have to solve the resulting system of linear equations and to determine \( c_i \) appearing in \( \phi(x,y) \). This approach requires a lot of error prone symbolic/numerical computations. For this reason, we feel that the present problem is an ideal field of application of computer algebra systems like Mathematica [3–7]. Other modern and powerful computer algebra systems, like Maple [8–11], can be used as well although Mathematica is more powerful and modern than Maple and, normally, much more rapid in floating-point computations.

We have called the approach of using computer algebra systems during the derivation of mixed symbolic/numerical results (exactly as is here the case) a SAN approach (from the words semi-analytical/numerical or symbolic–analytical–numerical). These results are not exact (as is here the situation with the Rayleigh–Ritz method) and, moreover, they require a lot of numerical computations together with symbolic computations. Therefore, the obtained results are quite general (since they include symbolic parameters), not restricted to constant (purely numerical) values of the geometric, loading and material parameters of an elasticity problem. This approach, although obvious as a principle, seems not to have been extensively used in practice contrary to the closed-form evaluation of quantities of interest in elasticity problems. At this point we can make reference to the recent results by Beltzer [12, 13] and the author [14–27], where additional references and sufficient details on the use of computer algebra systems in applied mechanics and related SAN computations can be found. Moreover, Beltzer [13, Section 3.7, pp. 99–101] applied the Rayleigh–Ritz method to the problem of a beam on an elastic foundation by using Macsyma.

Finally, the recent interest in using computer algebra systems in applied mechanics problems is reflected in the publication of two related books by the American Society of Mechanical Engineers (ASME) [28, 29] including the papers presented to ASME meetings with sessions on symbolic
computations. We hope that the present results with Mathematica [3–7] will further extend this
tendency and we feel that the classical plane elasticity problem (together with the Rayleigh–Ritz
method) is really an interesting area of application of the SAN approach with Mathematica, unfortunately,
to the best of our present knowledge, not having been used in computer algebra systems
applications up to now.

2. An application

We consider the classical problem of a rectangular plane isotropic elastic medium $D$ of dimen-
sions $2a$ by $2b$ with $-a \leq x \leq a$ and $-b \leq y \leq b$. This medium is assumed loaded only along two
of its edges, $x = \pm a$, with a tensile loading $\sigma_x$ of parabolic intensity given by

$$\sigma_x = S \left(1 - \frac{y^2}{b^2}\right), \quad (2)$$

where $S$ is a loading-intensity symbolic coefficient.

This problem is a classical one for the application of the Rayleigh–Ritz method and is reported
(as an application of this method) both by Timoshenko and Goodier [1] and by Sokolnikoff [2]. The
unknown Airy stress function $\phi(x,y)$ is sought in the following series form (where the boundary
conditions, especially in Eq. (2), have been already taken into consideration) [1]:

$$\phi(x,y) = \frac{1}{2} Sy^2 \left(1 - \frac{1}{6} \frac{y^2}{b^2}\right) + (x^2 - a^2)^2 (y^2 - b^2)^2 \times \left(c_1 + c_2 x^2 + c_3 y^2 + c_4 x^4 + c_5 y^4 + c_6 x^2 y^2 + \ldots\right). \quad (3)$$

The symmetry in geometry and loading has led to the appearance of only even powers in the vari-
ables $x$ and $y$ in the above expression of $\phi(x,y)$.

For the determination of the unknown coefficients $c_i$ we prepared a simple Mathematica procedure consisting of just few lines. (We have used version 1.2 of Mathematica available to us at this
moment.) This procedure, called plane, has the following form (with the Mathematica comments
included in it):

```mathematica
plane:=Block[{}
, (* the assumed form of \(\phi(x,y)\) inside \(D\) *)
  phi=(1/2)*S*y^2*(1-(1/6)*y^2/b^2)+(x^2-a^2)^2*(y^2-b^2)^2
  *(c[1]+c[2]*x^2+c[3]*y^2);
  (* the formulae for the stress components *)
  sx=D[phi,{y,2}]; sy=D[phi,{x,2}]; sxy=-D[phi,x,y];
  (* evaluation of the fundamental quantity \(V\) on \(D\) *)
  V=Simplify[Integrate[sx^2+sy^2+2*sxy^2,
  {x,-a,a}
  ,{y,-b,b}]];
  (* construction and solution of the linear equations *)
  Solution=Solve[Table[D[V,c[i]]==0,{i,3}],Table[c[i],{i,3}]];
  (* the algebraic rules for the final results for \(c_i\) *)
  ar=Solution[[1]];
  (* the final results for \(c_i\) from the above solution *)
  Do[c[i]=Together[c[i]/.ar]},{i,3}];
  (* end of the Mathematica procedure *)
]```
In the above procedure, plane, we have used three unknown SAN coefficients \( c_i \) \((i = 1, 2, 3)\). The use of more (or even less) such coefficients is quite trivial. Moreover, we have explicitly used the formulae for the stress components \([1]\) \( \sigma_x, \sigma_y \) and \( \sigma_{xy} \) in the same procedure in order both to get simpler formulae and, simultaneously, to show the natural way that Eq. (1) results from the strain energy density in the elastic region \( D \) \([1]\).

Finally, the Mathematica commands having been used in the above plane procedure are quite clear and can be found in detail in the related manual and books \([3–7]\). More explicitly, \( D \) is used for ordinary or partial differentiation, Integrate for single or multiple integration, Simplify for algebraic simplification, Solve for the solution of linear or nonlinear equations, Table for the formation of a list (in our case of the equations and the corresponding unknowns \( c_i, i = 1, 2, 3 \)) and Together for the formation of a single fraction instead of a sum of fractions. Finally, the Block command is normally used in Mathematica procedures in version 1.2 \([4]\).

3. SAN results

We display below the obtained SAN results from the above Mathematica procedure plane. At first, we have used only \( n = 1 \) coefficient, \( c_1 \), in plane, that is, we assumed \( c_i (i > 1) \) in Eq. (3) to be equal to zero. Then we find the following formula for \( c_1 \):

\[
c_1 = \frac{50176 S a^5 b^3}{458752 a^9 b^5 + 262144 a^7 b^7 + 458752 a^5 b^9}.
\]

(4)

In the special case of a square elastic medium \( D \) (that is, with \( a = b \)), Eq. (4) takes the simpler form

\[
c_1 = \frac{49 S}{1152 a^6} = 0.0425347 \frac{S}{a^6},
\]

(5)

which is in complete agreement with the related result in the literature \([1, 2]\).

Much more accurate SAN results can be obtained for \( n = 3 \) coefficients, \( c_1, c_2 \) and \( c_3 \) exactly as in the above-displayed procedure. These results are presented below:

\[
c_1 = S(110110 a^8 + 729729 a^6 b^2 + 5714863 a^4 b^4 + 729729 a^2 b^6 + 110110 b^8) / d,
\]

\[
c_2 = S(715715 a^8 + 1236235 a^6 b^2 + 170170 a^4 b^4 + 22022 b^6) / d,
\]

\[
c_3 = S(22022 a^8 + 170170 a^4 b^2 + 1236235 a^2 b^4 + 715715 b^6) / d
\]

(6)

with the common denominator \( d \) in these equations given by

\[
d = 1601600 a^{12} b^2 + 8303360 a^{10} b^4 + 58367872 a^8 b^6 + 46466816 a^6 b^8
\]

\[+ 58367872 a^4 b^{10} + 8303360 a^2 b^{12} + 1601600 b^{14}.\]

(7)

Special cases can be considered again as well. For example, for \( a = b \) we get from Eqs. (6) and (7)

\[
c_1 = \frac{0.0404046 S}{a^6}, \quad c_2 = c_3 = \frac{0.0117158 S}{a^8}.
\]

(8)

Similarly, for \( a = 2b \) we find

\[
c_1 = \frac{0.319283 S}{a^6}, \quad c_2 = \frac{0.499883 S}{a^8}, \quad c_3 = \frac{0.0738478 S}{a^8}.
\]

(9)

These values for \( c_i (i = 1, 2, 3) \) were found to be in agreement with the corresponding results displayed (in these special cases) by Timoshenko and Goodier \([1]\). Nevertheless, no general formulae
for these coefficients (for arbitrary values of the dimensions $a$ and $b$ of the elastic region $D$) are
given by these authors probably because of the great computational effort required during the SAN
solution (6) and (7) of the above linear equations. This seems to be a main reason that computer
algebra systems are very welcome in applications like the present one.

4. Discussion–generalizations

The above SAN results constitute the exact solution of the corresponding system of linear equa-
tions although this system is an approximation to the real elasticity problem and the corresponding
biharmonic equation. The computations having led to Eqs. (6) and (7) are very difficult to be per-
formed without the valuable help of the computer, which seems to be an indispensable tool in the
present, similar and more difficult cases. (In fact, here we have used an 80386/80387 MS-DOS
microcomputer at 20 MHz.) Evidently, even more accurate results can be obtained, of course, by
using higher values of the number $n$ of coefficients in the (finally) finite series (3) for the Airy stress
function $\phi(x,y)$. This will require much more computational effort by the computer, that is, more
time and more RAM and disk memory too. On the other hand, the obtained exact formulae will be
very accurate and, at the same time, complicated as well.

For this reason, an alternative possibility is to solve just approximately (and not exactly) the
system of linear equations. One such approach is to use the ratio $r = a/b$ of the dimensions of
the elastic medium $D$ and to find the unknown coefficients $c_i$ of $\phi(x,y)$ in Eq. (3) as Taylor series
of $r$ (with an appropriate number of terms). This approach has been already used in other applied
mechanics problems by the author [14, 23–25, 27] and can be considered now as a standard related
technique. Probably, other series expansions, like Chebyshev series [18] and minimax series [26]
could be used as well although the Taylor series seem to be the simplest possible ones.

Of course, the present approach for plane elasticity, based on the classical Rayleigh–Ritz method
and employing the powerful computer algebra system Mathematica [3–7], can be efficiently used
in a variety of more complicated plane elasticity problems, which can be found in books on elastic-
ity or even in the everyday practice of the engineer. The necessary modifications of the above
plane procedure will be rather simple. Moreover, obviously, several additional applied mechanics
problems treated by the Rayleigh–Ritz method and analogous methods can also be solved by appro-
priately modifying the present results. The generalization to the case of more than few nontrivial
symbolic parameters is also possible provided that multivariable Taylor series expansions will be
used in the SAN results. This is completely possible in Mathematica [3–7] and in Maple V [8–11].

As was already mentioned in the Introduction, Mathematica [3–7] is the most modern and
powerful computer algebra system for SAN computations like the present ones. Therefore, we
encourage the use of this C-based system instead of other LISP-based systems like Reduce and
Macsyma. (In fact, Beltzer [12, 13] used only Macsyma in his closed-form and SAN computations.)
Of course, Maple V [8–11] is also a C-based and very powerful computer algebra system but it is
less efficient than Mathematica in floating point computations requiring a special command,
evalhf, for the direct use of the computer hardware own algorithms in floating point computations.
Unfortunately, this command does not apply to mixed (SAN) computations, but only to purely
numerical computations.

In any case, it is believed that the SAN approach will become a classical approach for the
solution of elasticity and additional applied mechanics problems in the near future. Similarly,
computer algebra techniques and the related software will also continue gaining popularity among
researchers and engineers in applied mechanics. The appearance of two recent books [28, 29] with a
large number of papers in this area and two review papers [12, 30] in the same area seems to be also
an indication of an expected (at least by this author) boom in this kind of results in the near future.
The extremely rapid increase in power of modern microcomputers and workstations (together with the simultaneous decrease of their prices) also supports the aforementioned prediction, which may be also reflected in future HSTAM and GRACM national congresses on applied and computational mechanics.

References


3All the links (external links in blue) in this section were added by the author on 6 January 2018 for the online publication of this technical report. Moreover, in References [8, 11, 24–27, 30] final publication details were also added.


