Symbolic computations for the approximate solution of singular integral equations: application to a crack problem

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Abstract We propose the application of symbolic SAN (semi-analytical–numerical) computations to the numerical solution of SIEs (singular integral equations), which are the BIEs (boundary integral equations) for crack problems in plane and antiplane, isotropic and anisotropic elasticity. The case of a periodic array of collinear cracks (with a variable distance of the cracks) together with the modified Gauss–Chebyshev method (also based on the natural interpolation/extrapolation formula) for the numerical solution of SIEs are used for the illustration of the proposed approach. The obtained SAN results are seen to be very good approximations of the analytical exact results even for a very small number of nodes in the modified Gauss–Chebyshev method. The computer algebra system Derive has been used for the derivation of the present SAN results.

Keywords Boundary integral equations · Computer algebra · Symbolic computations · Semi-analytical–numerical computations · Singular integral equations · Gauss–Chebyshev method · Natural interpolation/extrapolation · Plane elasticity · Isotropic elasticity · Fracture mechanics · Cracks · Periodic array of collinear cracks

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1. Introduction

The BIE (boundary integral equation) method is one of the most powerful methods for the formulation of elasticity and many other problems in applied mechanics. The resulting equations are generally solved by the BEM (boundary element method) [1]. In crack problems, the BIE method reduces such problems to SIEs (singular integral equations) [2]. The modified Gauss–Chebyshev method, based also on the natural interpolation/extrapolation formula [3, 4], seems to be the most efficient method for the numerical solution of SIEs appearing in ordinary crack problems.

1Both the internal and the external links (all appearing in blue) were added by the author on 23 January 2018 for the online publication of this technical report.
Numerical computations suffer from the fact that they are valid only for concrete geometry/loading conditions. Therefore, their results are not of general validity. On the contrary, computer algebra and especially symbolic SAN (semi-analytical–numerical) methods improve this performance of numerical methods since they permit the appearance of symbols in the final results, which are, in this way, of much more general validity than those obtained by purely numerical methods. The recent advent of very powerful personal microcomputers made SAN methods much more effective than before. Today the most popular computer algebra systems seem to be Derive, Macsyma, Maple, Mathematica and Reduce. A recent review of these systems can be found in Reference [5]. Here we have used Derive [6], the most elementary and friendly to the user of these systems, but all of the above systems (and some more) could also be effectively used.

The aim of this technical report is simply to illustrate the SAN approach in the solution of SIEs. We are unaware of applications of this computational approach to SIEs and BIEs in general. On the contrary, we are aware of several more or less recent publications in computational applied mechanics which are based on computer algebra software. Some related references are References [7–17].

2. Periodic array of collinear cracks

We consider the classical problem of a periodic array of collinear cracks in plane isotropic elasticity (see, e.g., [18]). The critical quantity in such a fracture mechanics problem is the mode I stress intensity factor \( K \) at the crack tips. Denoting by \( 2a \) the length of each crack and by \( b \) the period of the array, we have for \( K \) [18]

\[
K = \sigma \sqrt{\pi a k},
\]

where \( \sigma \) is the tensile loading intensity and \( k \) is the dimensionless stress intensity factor for our problem having been found to be determined by [18]

\[
k = \sqrt{\frac{\tan c}{c}}.
\]

Here \( c \) is a dimensionless variable equal to

\[
c = \frac{\pi a}{b}.
\]

We have used the TAYLOR command in Derive [6] in order to find the Maclaurin series (Taylor series at \( c = 0 \)) of \( \tan c \). Next, we have divided the result by \( c \) and, taking into account Eq. (2), we have used the same command for the square root of the last result. In this way, we obtained (by using Derive [6]) a Maclaurin series approximation to \( k \) in Eq. (2), i.e.

\[
k = 1 + \frac{1}{6} c^2 + \frac{19}{360} c^4 + \frac{55}{3024} c^6 + O(c^8).
\]

From Eq. (4) it is clear that this equation is sufficiently accurate for small and moderate values of the variable \( c \). Therefore, we can consider Eq. (4) as a very good approximation to Eq. (2).

3. Application of the SIE method

We consider the SIE method of the crack problem of the previous section, where, because of Eq. (1), we can assume the loading \( \sigma \) to be equal to 1. Then we find the SIE [2, pp. 269–276], [19]

\[
\frac{c}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \cot[c(t-x)] g(t) \, dt = 1.
\]

(5)
Here \( c \) was defined by Eq. (3) and \( g(t) \) is the unknown function in our SIE related to the deformed shape of the crack edges. The condition of single-valuedness of displacements,
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} g(t) \, dt = 0,
\]
accompanies Eq. (5).

By applying the modified Gauss–Chebyshev numerical method for the solution of SIEs [3] to Eqs. (5) and (6), we obtain
\[
g_n(x) = \frac{T_n(x)}{U_{n-1}(x)} \left\{ 1 - \frac{c}{n} \sum_{i=1}^{n} \cot(c(t_{in} - x))g_n(t_{in}) \right\}.
\]
Here \( g_n(x) \) denotes an approximation to \( g(x) \), due to the application of the Gauss–Chebyshev quadrature rule, \( T_n(x) \) and \( U_n(x) \) denote the classical Chebyshev polynomials of degree \( n \) of the first and the second kind, respectively, \( t_{in} \) are the nodes of the Gauss–Chebyshev quadrature rule given by
\[
t_{in} = \cos \left( \frac{(2i-1)\pi}{2n} \right), \quad i = 1, 2, \ldots, n,
\]
and, finally, \( g_n(t_{in}) \) are the values of \( g_n(x) \) at the nodes \( t_{in} \). These values are obtained from the solution of the following system of \( n \) linear algebraic equations:
\[
\frac{c}{n} \sum_{i=1}^{n} \cot\left[ c(t_{in} - x_{kn}) \right]g_n(t_{in}) = 1, \quad k = 1, 2, \ldots, n-1,
\]
\[
\frac{1}{n} \sum_{i=1}^{n} g_n(t_{in}) = 0,
\]
where the collocation points \( x_{kn} \) are given by [3]
\[
x_{kn} = \cos \left( \frac{k\pi}{n} \right), \quad k = 1, 2, \ldots, n-1.
\]

In our case, because of symmetry conditions, that is
\[
g_n(-t_{in}) = -g_n(t_{in}), \quad i = 1, 2, \ldots, n,
\]
the system of linear algebraic Eqs. (9) and (10) takes the following simpler form (here with an even number of nodes \( n \)):
\[
\frac{c}{n} \sum_{i=1}^{n/2} \left\{ \cot\left[ c(t_{in} - x_{kn}) \right] + \cot\left[ c(t_{in} + x_{kn}) \right] \right\}g_n(t_{in}) = 1, \quad k = 1, 2, \ldots, n/2.
\]

Moreover, the natural interpolation/extrapolation formula (7), valid for all the points of the crack (with the exception of the nodes \( t_{in} \) and the collocation points \( x_{kn} \)) reduces (because of Eqs. (12)) to
\[
g_n(x) = \frac{T_n(x)}{U_{n-1}(x)} \left\{ 1 - \frac{c}{n} \sum_{i=1}^{n/2} \left\{ \cot\left[ c(t_{in} - x) \right] + \cot\left[ c(t_{in} + x) \right] \right\}g_n(t_{in}) \right\}.
\]

In fracture mechanics, from the practical point of view we are interested in the stress intensity factor \( K \). The dimensionless stress intensity factor \( k \) at the crack tips results to be simply determined by [2, p. 273]
\[
k = g(1).
\]
Therefore, because of Eq. (14) and the properties of the Chebyshev polynomials, we easily find that the approximation \( k_n \) to \( k \) based on the approximation \( g_n(x) \) in Eq. (14) is given by

\[
k_n = \frac{1}{n} \left\{ 1 + \frac{c}{n} \sum_{i=1}^{n/2} \left( \cot[c(1-t_i)] - \cot[c(1+t_i)] \right) g_n(t_i) \right\}.
\]  

(16)

This is our final formula by application of the modified Gauss–Chebyshev method [3, 4, 19], where the nodes \( t_i \) are determined from Eqs. (8) and the values \( g_n(t_i) \) of the unknown function \( g_n(x) \) are determined from the system of linear algebraic Eqs. (13).

4. Illustration of the SAN approach

For the illustration of the SAN approach in the crack problem of the previous two sections, at first we consider the very simple case where \( n = 2 \). Then, because of Eqs. (8) and (11), we have

\[
t_{11} = \frac{1}{\sqrt{2}}, \quad x_{11} = 0
\]  

(17)

and Eqs. (13) take the simple form of a single equation, i.e.

\[c \cot \frac{c}{\sqrt{2}} g_2(t_{11}) = 1.\]  

(18)

Therefore,

\[g_2(t_{11}) = \frac{1}{c} \tan \frac{c}{\sqrt{2}}.\]  

(19)

Then Eq. (16) yields

\[
k_2 = \frac{1}{2} + \frac{c}{4} \left\{ \cot \left[ c \left( 1 - \frac{1}{\sqrt{2}} \right) \right] - \cot \left[ c \left( 1 + \frac{1}{\sqrt{2}} \right) \right] \right\} \frac{1}{c} \tan \frac{c}{\sqrt{2}}.
\]  

(20)

This is our final formula for the dimensionless stress intensity factor \( k_2 \approx k \) under consideration.

Frequently, it is advisable to have the Maclaurin series (equivalently, the Taylor series at \( c = 0 \)) of \( k_2 \) from Eq. (20). Moreover, in our case, this will permit us to find a simple expression (without trigonometric functions) for \( k_2 \). Next, it will be possible to compare this series with the exact series (4) obtained (by using Derive [6]) in Section 2.

The TAYLOR command of Derive [6] permitted us to find, quite easily, that

\[c \cot \left[ c \left( 1 - \frac{1}{\sqrt{2}} \right) \right] = 2 + \sqrt{2} + \left( \frac{\sqrt{2}}{6} - \frac{1}{3} \right) c^2 + \left( \frac{7\sqrt{2}}{180} - \frac{1}{18} \right) c^4 + O(c^6),\]  

(21)

\[c \cot \left[ c \left( 1 + \frac{1}{\sqrt{2}} \right) \right] = 2 - \sqrt{2} - \left( \frac{\sqrt{2}}{6} + \frac{1}{3} \right) c^2 - \left( \frac{7\sqrt{2}}{180} + \frac{1}{18} \right) c^4 + O(c^6).\]  

(22)

Then

\[
\frac{c}{4} \left\{ \cot \left[ c \left( 1 - \frac{1}{\sqrt{2}} \right) \right] - \cot \left[ c \left( 1 + \frac{1}{\sqrt{2}} \right) \right] \right\} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} c^2 + \frac{\sqrt{2}}{360} c^4 + O(c^6).
\]  

(23)

On the other hand, we also found that

\[g_2(t_{11}) = \frac{1}{c} \tan \frac{c}{\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} c^2 + \frac{\sqrt{2}}{60} c^4 + O(c^6).\]  

(24)

Then Eq. (20) yielded directly that

\[k_2 = 1 + \frac{1}{6} c^2 + \frac{1}{20} c^4 + O(c^6).\]  

(25)
By comparing Eqs. (4) and (25), we observe that

\[ k - k_2 = \frac{1}{360} c^4 + O(c^6) \]  
\[ (26) \]

and, in our opinion, this is a very good result as far as the approximation \( k_2 \) to \( k \) is concerned. We can also observe that all of these computations were made by the computer by using Derive [6].

An even better result for the approximation to \( k \) was obtained for \( n = 4 \) by using again the formulae of the previous section. In Fig. 1 (first part on this page, p. 5, and second part on the next page, p. 6) we display the corresponding results (equations (D1) to (D31)). These results were obtained by using Derive [6] in its printing format (with very little editing by us for a better appearance, but, of course, not on the formulae themselves; the tilde is used by Derive as a continuation sign inside a multiple-line formula).

More explicitly, in Fig. 1 (D1) is the Laurent series approximation of \( \cot y \) at \( y = 0 \) [20, p. 88]
with five nonzero terms. (D2–D3) and (D4–D5) are the values of the nodes \( t_m \) and the collocation points \( x_{kn} \), respectively, obtained from Eqs. (8) and (11) for \( n = 4 \), but required in Eqs. (13), (14) and (16) for \( i, k = 1, 2, \ldots, n/2 \) only. (D6), (D8), (D10) and (D12) are the formulae for the four coefficients \( a_{ik} \) in the left-hand side of the system of linear algebraic Eqs. (13) (in our case, two equations in two unknowns) and, further, (D7), (D9), (D11) and (D13) are the corresponding Maclaurin series with five nonzero terms and an accuracy of ten significant digits having been obtained by using (D1). (D14) and (D15) illustrate Derive's \texttt{SOLVE} command applied to the symbolic (formal) solution of a linear system of two equations in two unknowns. Although here this command is of marginal importance, this is not the case in more complicated systems of equations.

In (D16), (D18) and (D20), we give the formulae for the denominator and the two numerators in the solution (D15) of the aforementioned system of two linear algebraic equations in two unknowns having appeared in the present application by using Eqs. (13) as was already mentioned. In (D16), we observe the use of the \texttt{TAYLOR} command of Derive in order to keep the order of the Maclaurin series approximation (with the variable \( c \) given again by Eq. (3)) to \( m = 8 \). In (D17), (D19) and (D21), we display (always through Derive) the corresponding (to (D16), (D18) and (D20), respectively) Maclaurin series approximations once more with the desired accuracy.

**Fig. 1** (second part): SAN results from Derive for a periodic array of collinear cracks obtained by the modified Gauss–Chebyshev method and \( n = 4 \) nodes (results continued from the previous page, p. 5).
Next, in (D22) and (D23), we find (again by using the TAYLOR command of Derive) the Maclaurin series of the inverse of the denominator to be used directly in the numerators. This is really the case in (D24) to (D27), where the formulae (D15) for the solution of our system are used and the Maclaurin series of $g_n(t_m)$ in Eqs. (13) (in our case with $n = 4$ and with Eq. (12) taken into account) are obtained.

Finally, it is possible to use Eq. (16) in order to obtain the corresponding parametric expression for the approximation $k_4$ to $k$ (for $n = 4$). In Fig. 1, this is made in (D28) and (D29). We observe that (D28) is completely equivalent to Eq. (16) (for $n = 4$) although in Fig. 1 written in Derive’s output format. The exact Maclaurin series for $k$ is displayed in (D30) (or, equivalently, in Eq. (4)) and, in decimal notation, in (D31). By comparing (D29) and (D31), we directly observe that now

$$k - k_4 = O(e^8)$$

(27)

instead of $O(e^4)$ as has been the case in Section 4 for $n = 2$ and is clear from Eq. (26). We have not available a theoretical formula more accurate than Eq. (4) for $k$ in order to be able to compare $k_4$ with the corresponding exact value of $k$. In any case, the above result, Eq. (27), is an excellent one and it clearly shows the efficiency of the present SAN approach.

5. Discussion

The above symbolic computations were made on a Tulip PC compact 2 IBM PC/XT compatible personal microcomputer without any difficulty. Of course, an 80386 microcomputer would be a more effective choice, but no such computer is presently at our disposition privately or in our Department at the University of Patras. On the other hand, it would also be advantageous to use a computer algebra system with programming capabilities such as Macsyma, Maple, Mathematica and Reduce [5]. This would make the symbolic computations of Fig. 1 easier. Unfortunately, no such facility was found possible in our Department as well as in the Computing Centre of the University of Patras at the present time.

Another interesting possibility is the use of fast computer algebra systems supporting mathematical coprocessors for microcomputers (such as Macsyma or, better, Mathematica); this is absolutely necessary in bulky computational mechanics applications. (It was announced that Mathematica, beyond its built-in very high speed in floating point computations, supports both Intel’s and Weitek’s mathematical coprocessors as well.) Of course, the use of workstations (instead of microcomputers) is also a very interesting possibility.

In spite of the lack of the above facilities, we feel that we have already been able to show the effectiveness of the SAN approach in the problem under consideration and we hope that this approach will become popular in the future in computational mechanics applications, more explicitly, in the symbolic (formal) solution of BIEs and SIEs as was explained in Section 1. Future will show whether we have been right or wrong in this opinion of us.

References


