Locating inclusions of the same material in finite plane isotropic elastic media by using complex path-independent integrals

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Abstract  The method of complex path-independent integrals on a closed contour is used for the location of an inclusion (of arbitrary but known shape) of the same material with the matrix and welded with the matrix in plane isotropic elasticity for a finite medium. Only the position of the inclusion and the external loading are not known in advance. The first complex potential of Kolosov–Muskhelishvili (or one of its two first derivatives) is used, together with optical methods for its evaluation, on the aforementioned contour. Beyond the location of the inclusion, a variety of generalizations of the proposed technique as well as a long discussion are also included.

Keywords  Inclusions · Location of inclusions · Plane isotropic elasticity · Complex path-independent integrals · Closed contours · Contour integrals · Cauchy-type integrals · Complex potentials · Complex analysis · Analytic functions · Holomorphic functions · Residues

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1. Introduction

We consider the problem already stated above. In the subsequent sections, we will study (i) The statement of the problem and its solution if the position of the inclusion is known in advance [1]. (ii) The fundamentals and a discussion of the powerful method of complex path-independent integrals both from the analytic functions point of view as well as from the plane isotropic elasticity point of view. (iii) The formulation of the problem by using the first complex potential $\phi(z)$ of Kolosov–Muskhelishvili or its derivative $\Phi(z)$ or its second derivative $\Phi'(z)$ studied in detail in Reference [1]. (iv) The solution of the problem of location of the inclusion by using the method of complex path-independent integrals on a contour $C$ surrounding the inclusion. (v) The conclusion of this solution as far as the position of the inclusion is concerned together with a variety of comments on additional information about the inclusion, possibilities of generalizations and a discussion. We believe that the results of this technical report, beyond their own interest, may find a wide range of generalizations and practical applications in nondestructive testing of elastic media.

2. Statement of the problem

In this technical report, we revisit a classical application of the problem of linear relationship (more commonly known as Riemann problem or Hilbert problem or even Riemann–Hilbert problem): that of a finite plane isotropic elastic medium $S_0$ (its boundary denoted by $L_0$) with one inclusion $S$ (its boundary denoted by $L$) of the same material with the matrix (Fig. 1). This problem was solved by Sherman, whose results are reported by Muskhelishvili [1]. It is supposed that the boundary of the inclusion $L$ differs slightly, in the unstrained state, from that of the corresponding hole of the matrix, in such a way that complete contact (without any gap) has been achieved on $L$ and, moreover, that the matrix and the inclusion are welded together on $L$ or restrained from frictional forces from sliding relatively to each other. These assumptions are the same as in Reference [1].
Both $L_0$ and $L$ are assumed sectionally smooth simple closed contours. Moreover, it is assumed that the discontinuities in the displacements $u$ and $v$ for a passage through $L$ are given functions: $u^+(t) - u^-(t) \equiv g_1(t)$, $v^+(t) - v^-(t) \equiv g_2(t)$, $t \in L$, where $t = x + iy$ are the points of $L$ (in complex notation). Both $g_1(t)$ and $g_2(t)$ depend on the shape of the inclusion (relatively to the matrix) before deformation and on the method by which the edges of the inclusion and of the matrix were brought into contact before welding occurred [1]. Furthermore, all of the above information is assumed known in advance. The solution of this problem (under somewhat more general assumptions) is reported in detail by Muskheilishvili [1]. More explicitly, the first complex potential $\phi(z)$ of Kolosov–Muskheilishvili is given by [1]

$$\phi(z) = \phi_0(z) + \phi^*(z), \quad \phi^*(z) = \frac{\mu}{\pi i (\kappa + 1)} \int_L \frac{g(t)}{t - z} \, dt, \quad g(t) = g_1(t) + ig_2(t),$$  \hspace{1cm} (1)

where $\kappa$ and $\mu$ are the related constants of the elastic material (the Muskheilishvili constant and the shear modulus, respectively) and $\phi_0(z)$ is an analytic (holomorphic) function both in $S_0$ and in $S$ depending on the loading of the elastic medium (mainly on $L_0$) assumed here not known. The reason for reconsidering this problem (and reporting well-known results above) is that we further wish to apply the concept of complex path-independent integrals in a completely original, new class of applications. We make some remarks on this concept in the next section.

3. On complex path-independent integrals

Classical results on complex path-independent integrals in the theory of analytic functions are well known since last century and reported in every book on complex analysis; see, e.g., Churchill, Brown and Verhey [2]. The equally well known classical Cauchy’s theorem and Cauchy’s residue theorem are two of their important applications.

On the contrary, in plane elasticity it seems that the concept of complex path-independent integrals was used for the first time by Budiansky and Rice [3], rather not as an efficient tool for the treatment of plane elasticity problems, but just as a way of alternative expressions of few path-independent integrals in plane elasticity. Next, the author proposed the use in elasticity of the concept of complex path-independent integrals as a tool for the estimation of stress intensity factors. (An infinity of such integrals can be constructed in this way.) His results are reported by Theocaris and Ioakimidis [4]. In the same paper, two particular explicit experimental methods were reported for the determination of the second derivative $\Phi'(z) = \phi''(z)$ of $\phi(z)$ (with $\Phi(z) \equiv \phi(z)$) in engineering applications. The first method uses small reflective mirrors on a contour $C$ (where the path-independent integrals are evaluated), whereas the second method uses, instead, an appropriate mesh. (It seems that the first method is superior if $C$ lies inside the elastic medium $S_0$, whereas the second method is superior if $C$ coincides with the boundary $L_0$ of $S_0$.) Both of these methods have been effectively used in practice and they are essentially completely equivalent. These remarks concern just the potential reader of the present technical report whose sole aim is its experimental use. This is not the author’s sole aim; his aim is to show the efficiency of the concept of complex path-independent integrals from the theoretical than from the practical point of view in elasticity and, more generally, in applied mechanics and engineering.

The aforementioned general results of the author (Theocaris and Ioakimidis [4]) were generalized and repeatedly used by the same author and other authors. The related references are reported in two recent papers by the author (Ioakimidis [5, 6]), of course, together with additional related results. But this is not so important; what is important is the generalization of the methods of locating zeros and poles of analytic functions in the complex plane to locating singular points or cracks in plane elasticity always by the method of complex path-independent integrals together with the Cauchy theorem and the Cauchy residue theorem in complex analysis.
In fact, complex path-independent integrals in the theory of analytic functions have been repeatedly used for the location of zeros of analytic and sectionally analytic functions in a variety of more or less ingenious methods. Most of the related results are reported in a recent review paper by the author (Ioakimidis [7]). In elasticity problems, the case of poles of meromorphic (analytic with poles) functions is more important. It seems that this case was considered at first by Abd-Elall, Delves and Reid [8]. A generalization of these results is due to the author (Ioakimidis [9]). Additional results are also available. In elasticity problems, the location of a concentrated force or an edge dislocation is exactly that of the location of a pole of a meromorphic function if the complex potential \( \Phi(z) = \phi(z) \) is used (Muskhelishvili [1]). We give just one reference (Ioakimidis [10]) concerning a straight crack under special loading conditions and the determination of its tips by using complex path-independent integrals. Contrary to the previous original results of the author (Theocaris and Ioakimidis [10]), which have been repeatedly generalized, his present results seem still remaining a field of exclusive research by the author.

In an attempt to generalize the applications of the technique of complex path-independent integrals to plane elasticity for the location of singular points, cracks and other singular shapes, the author proposed (very recently) the application of the aforementioned technique to the location of circular holes and inclusions (Ioakimidis [6]). Here we go on by considering the aforementioned and completely described problem of the previous section. We will solve this problem by the method of complex path-independent integrals with the belief that our solution will be sufficiently clear and understandable to the reader so that it can become useful even for quite practical engineering and industrial applications. (These rather long introductory sections aimed at this purpose.) The critical point in the solution is the reduction of the problem of location of an inclusion of arbitrary shape (welded together with the matrix and under completely arbitrary loading) to that of locating the pole of a meromorphic function. (The paper by Abd-Elall, Delves and Reid [8] has been the fundamental one.) Now we proceed to our solution.

4. Formulation of the problem

We reconsider the problem of Section 2 (Muskhelishvili [1]) with the aim to locate the inclusion under arbitrary loading conditions. The shape of the inclusion (bounded by \( L \)) and the complex function \( g(t) \) (defined in Eq. (1)) are assumed known in advance. The whole situation is clear from Fig. 1, where \( C \) is the curve on which the complex path-independent integrals will be evaluated. We assume that we have information about \( \phi(z) \) and/or its derivatives, \( \Phi(z) \) or \( \Phi'(z) \), only on \( C \). No information about the loading on \( L_0 \) is available. (\( C \) and \( L_0 \) may coincide, contrary to Fig. 1, as was already mentioned in the previous section.) We consider two separate coordinate systems. One of these, \( Oxy \), is defined by us (it is attached to \( C \), which is a known curve, or to \( L_0 \)), whereas the second of these, \( K\xi\eta \), is attached to the inclusion \( S \) (probably with its centre \( K \) inside this inclusion). The axes of these coordinate systems are assumed parallel. The corresponding complex variables are denoted by \( z = x + iy \) and \( \xi = \xi + i\eta \), respectively. For \( L \) they are denoted by \( t \) (with respect to \( Oxy \)) and \( \tau \) (with respect to \( K\xi\eta \)), respectively (Fig. 1). Clearly, \( g(t) \) in Eq. (1) can also be written as \( g^*(\tau) \), since at any point of \( L \) it is known in advance (independently of the position of \( L \) with respect to \( C \) and \( L_0 \) and of the loading conditions). Therefore, the whole problem reduces to that of locating \( K \) (with respect to \( Oxy \)). The position of \( K \) with respect to the primary coordinate system \( Oxy \) is defined by \( a = a_1 + ia_2 \) (Fig. 1). We will determine \( a \) by using information gathered on \( C \) only. This will permit the location of the inclusion \( S \).

We rewrite Eq. (1) and its two first derivatives (in our case useful only in \( S_0 \))

\[
\phi(z) = \phi_0(z) + \frac{\mu}{\pi i(\kappa + 1)} \int_L \frac{g(t)}{t - z} \, dt,
\]
\[ \phi'(z) = \Phi(z) = \phi'_0(z) + \frac{\mu}{\pi(i(k + 1))} \int_L \frac{g(t)}{(t-z)^2} dt = \phi'_0(z) + \frac{\mu}{\pi(i(k + 1))} \int_L \frac{g'(t)}{t-z} dt, \]  
\[ \phi''(z) = \Phi'(z) = \phi''_0(z) + \frac{2\mu}{\pi(i(k + 1))} \int_L \frac{g(t)}{(t-z)^3} dt = \phi''_0(z) + \frac{\mu}{\pi(i(k + 1))} \int_L \frac{g''(t)}{t-z} dt, \]
where the classical results of integration by parts (Gakhov [11]) for Cauchy-type integrals on closed contours \(C\) have been taken into account in the last parts of Eqs. (3) and (4). Moreover, the densities in the Cauchy-type integrals in Eqs. (2) to (4) are assumed, for convenience, to be Hölder-continuous functions. Of course, we assume that \(\phi^*(z)\) in Eq. (1) does not vanish identically as may happen in an extremely exceptional case.

Due to the Cauchy theorem in classical complex analysis, for an analytic function \(F(z)\) on \(C\) and inside \(C\) (Churchill, Brown and Verhey [2]) we have
\[ \int_C F(z) dz = 0. \] (5)
In our problem, this is the case with \(\phi_0(z), \phi'_0(z)\) and \(\phi''_0(z)\). Therefore, since \(g(t) \equiv g^*(\tau)\) is a known function in our case (independent of the position of the inclusion \(S\)) and \(t = \tau + a\) (Fig. 1), it is completely clear from Eqs. (2), (3) and (4) that we have to find a by using the known values (on \(C\)) of an unknown function \(f(z)\) of one of the following forms (with \(F(z)\) an analytic function on \(C\) and in the domain bounded by \(C\)):
\[ f(z) = F(z) + \frac{1}{2\pi i} \int_C \frac{h(\tau)}{\tau - (z-a)} d\tau, \] (6)
\[ f(z) = F(z) + \frac{1}{2\pi i} \int_C \frac{h(\tau)}{[\tau - (z-a)]^2} d\tau, \] (7)
\[ f(z) = F(z) + \frac{1}{\pi i} \int_C \frac{h(\tau)}{[\tau - (z-a)]^3} d\tau, \] (8)
(and only one of these forms). For example, the first form, Eq. (6), appears when we use Eq. (2) or the second of Eqs. (3) or the second of Eqs. (4) with
\[ F(z) = \phi_0(z), \quad \phi'_0(z) \quad \text{and} \quad \phi''_0(z), \] (9)
respectively, and
\[ h(\tau) = \frac{2\mu}{\kappa + 1} g^*(\tau), \quad \frac{2\mu}{\kappa + 1} g^*(\tau) \quad \text{and} \quad \frac{2\mu}{\kappa + 1} g^*(\tau), \] (10)
respectively. Similarly, the second form, Eq. (7), appears when we use the first of Eqs. (3) and the third form, Eq. (8), appears when we use the first of Eqs. (4), together with the first of Eqs. (10) in both these cases. (Usually, as was reported in brief in the previous section, we have available, by optical measurements, \(\Phi'(z)\) on \(C\). We are now ready to proceed to the solution of our problem, that is to the determination of \(a\), from Eq. (6) or Eq. (7) or Eq. (8). Due to Eq. (5), we ignore \(F(z)\) in the complex path-independent integrals on the closed contour \(C\) surrounding the inclusion \(S\).  

5. Solution of the problem

We take into account the fundamental extremely elementary formula of geometric series [2]
\[ \frac{1}{1-w} = \sum_{k=0}^{\infty} w^k, \quad |w| < 1, \] (11)
and its first two derivatives
\[
\frac{1}{(1-w)^2} = \sum_{k=0}^{\infty} k w^{k-1}, \quad \frac{2}{(1-w)^3} = \sum_{k=0}^{\infty} k(k-1) w^{k-2}, \quad |w| < 1. \tag{12}
\]

In our case, using these equations, we have the well-known results
\[
\frac{1}{\tau - (z-a)} = -\sum_{k=0}^{\infty} \frac{\tau^k}{(z-a)^{k+1}}, \tag{13}
\]
\[
\frac{1}{[\tau - (z-a)]^2} = +\sum_{k=0}^{\infty} \frac{(k+1) \tau^k}{(z-a)^{k+2}}, \tag{14}
\]
\[
\frac{2}{[\tau - (z-a)]^3} = -\sum_{k=0}^{\infty} \frac{(k+1)(k+2) \tau^k}{(z-a)^{k+3}}, \tag{15}
\]
always under the restriction (assumed fulfilled and really fulfilled in practice since \(C\) lies somewhat away from \(L\))
\[
|\tau| < |\zeta| = |z-a| \tag{16}
\]
for all points \(\tau\) of \(L\) and for all points \(\zeta = z-a\) of \(C\).

Having available in advance \(h(\tau)\) in Eqs. (6), (7) and (8) (as was already explained under the assumptions of this technical report), we have also available (analytically or after numerical integration) the quantities
\[
A_k = -\int_L \tau^k h(\tau) \, d\tau, \quad k = 0, 1, \ldots, \tag{17}
\]
where, of course, the integrals are assumed evaluated in the positive, anticlockwise, sense for \(L\); the same is assumed for \(C\) as well. Moreover, having available (experimentally or theoretically or numerically) the values of \(f(z)\) on \(C\), we construct the following complex path-independent integrals on this contour:
\[
I_m = \int_C z^m f(z) \, dz, \quad m = 0, 1, \ldots. \tag{18}
\]
In practice, the values of \(f(z)\) are available at a concrete set of points on \(C\) and an appropriate numerical integration rule, usually the trapezoidal or an equispaced one, is used for the approximate determination of \(I_m\). We will not consider the resulting related errors either due to experimental techniques, beyond the comments we have already made, or to numerical integration although the latter are decreased by increasing the number of nodes in numerical integration.

Now, at first, by using Eqs. (13), (14) and (15), we rewrite Eqs. (6), (7) and (8) as
\[
f(z) = F(z) + \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{A_k}{(z-a)^{k+1}}, \tag{19}
\]
\[
f(z) = F(z) - \frac{1}{2\pi i} \sum_{k=0}^{\infty} (k+1) \frac{A_k}{(z-a)^{k+2}}, \tag{20}
\]
\[
f(z) = F(z) + \frac{1}{2\pi i} \sum_{k=0}^{\infty} (k+1)(k+2) \frac{A_k}{(z-a)^{k+3}}, \tag{21}
\]
respectively, where, of course, the definitions (17) of the quantities \(A_k\) have been taken into account. Moreover, obviously, Eqs. (20) and (21) result with one and two differentiations, respectively, of Eq. (19)
Next, by taking into account Eq. (5) as well as the classical Cauchy residue theorem of complex analysis (Churchill, Brown and Verhey [2]), we obtain from Eqs. (18) because of Eq. (19) (in the case when Eq. (6) is of interest)

\[
\begin{align*}
I_0 &= A_0, \\
I_1 &= aA_0 + A_1, \\
I_2 &= a^2A_0 + 2aA_1 + A_2, \\
I_3 &= a^3A_0 + 3a^2A_1 + 3aA_2 + A_3
\end{align*}
\]

and so on since \( z = a + (z - a) \). In an inverse way, we obtain from the Cauchy residue theorem

\[
\int_C (z-a)^m f(z) \, dz = A_m, \quad m = 0, 1, \ldots,
\]

by consideration of the complex path-independent integrals of its left-hand side. From Eqs. (23) we directly find (because of Eqs. (18)) that

\[
\begin{align*}
A_0 &= I_0, \\
A_1 &= I_1 - aI_0, \\
A_2 &= I_2 - 2aI_1 + a^2I_0, \\
A_3 &= I_3 - 3aI_2 + 3a^2I_1 - a^3I_0
\end{align*}
\]

and so on. A simple table of the classical binomial coefficients permits us to find a series of equations beyond Eqs. (22) and (24). Of course, evidently, Eqs. (22) and (24) are compatible, that is each set of these equations results directly from the other.

The above results can be modified to apply when we use Eq. (7) and, equivalently, Eq. (20), or Eq. (8) and, equivalently, Eq. (21). The sets of related formulas, corresponding to Eqs. (22) and (24), are so easily obtainable that we will not waste space to list these here. Nevertheless, it can be remarked from Eq. (20) that \( I_0 = 0 \) in this case and from Eq. (21) that both \( I_0 \) and \( I_1 = 0 \) in this case. These particular complex path-independent integrals can be used in practice under the above assumptions only for the verification of the obtained results for \( f(z) \) on \( C \) and of the efficiency of the applied numerical integration rule.

6. The conclusions and discussion

In the case of Eq. (6), valid in all three cases of Eqs. (2), (3) and (4) (the cases of Eqs. (7) and (8) are completely analogous as was just mentioned), we obtain from the second of Eqs. (22) that

\[
a = \frac{I_1 - A_1}{A_0}.
\]

Similarly, we obtain from the second of Eqs. (24)

\[
a = \frac{I_1 - A_1}{I_0}.
\]

Because of the first of Eqs. (22) (or the first of Eqs. (24)), these formulas coincide. Therefore, the position of the point \( K \), attached to the inclusion \( S \) and essentially completely defining the position of this inclusion, has been determined and our original problem has been solved. Because of experimental and/or numerical errors, the verification of one or two more of Eqs. (22) or (24) is recommended in a practical engineering environment.
On the other hand, using just Eqs. (24), we observe that we have an infinite set of equations determining the quantities $A_k$, defined by Eqs. (17), on the basis of the complex path-independent integrals $I_m$ based on information gathered exclusively on $C$, frequently far away from the inclusion $S$. Each one of these equations can be considered (at least in principle) permitting us to determine one quantity of interest beyond $a$ already determined from Eq. (25) or Eq. (26). In this way, we can assume not only the location (essentially expressed by $a$), but also a long list of parameters about our inclusion $S$ as unknowns (for example, the radius in the case of a circular inclusion, the lengths of the axes in the case of an elliptical inclusion and so on) as far as the dimensions of the inclusion are concerned and, in an analogous way, related quantities concerning the function $g(r)$, that is, essentially, the way of welding between the matrix and the inclusion. We realize that in such cases the computational work becomes more complicated: a system of nonlinear equations has usually to be solved for the determination of the unknown quantities and, moreover, the obtained results will probably be less accurate than expected. For this reason, we propose the extension of the present technique of locating $S$ to the determination of as few as possible additional quantities concerning this inclusion together with the verification of the finally obtained results by using additional equations from the set of Eqs. (24) in the case of Eq. (6). Our previous results (Ioakimidis [5]), although concerning cracks and, moreover, not their location, nevertheless, to some extent they seem to be of interest with respect to the remarks of this paragraph.

Finally, we can add that the location of the inclusion of this technical report, by using path-independent integrals, may be considered as the beginning of a theoretical and, under appropriate conditions, practical method of nondestructive testing in elasticity problems in regions of lack of accessibility especially under loading, working, conditions. This fact seems to be of particular interest especially in three-dimensional elasticity as well as in dynamic elasticity for the location of cracks, holes, inclusions, etc. Obviously, the path-independent integrals will not be complex in these cases although hyperanalytic functions with a well-established theory in mathematics may be used in companion with the corresponding real path-independent integrals available in the literature. The required theoretical effort does not seem to be too great, but the author is not aware of any related literature (location of cracks, holes, inclusions, etc.) by using path-independent integrals although he cannot assure at all that this is or might be the case. We conclude this technical report with this remark hoping that the results of this report will prove useful in a real engineering environment in the future.

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