Methods of numerical solution of singular integral equations with Cauchy-type kernels

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Abstract The direct methods are the most efficient methods of numerical solution of singular integral equations with Cauchy-type kernels, which appear in several physics and engineering problems. Here the fundamental already available recent results concerning these methods, which were found mainly during the last fifteen years, are presented in brief. More explicitly, after an introduction to singular integral equations both of the second kind (with variable or constant coefficients) and of the first kind, the four best known direct methods of numerical solution of singular integral equations, i.e. (i) the Galerkin method, (ii) the collocation method, (iii) the quadrature–collocation method and (iv) the quadrature method, are described in some detail. The numerical integration rules for Cauchy-type principal value integrals used in the quadrature–collocation and the quadrature methods are also presented. Moreover, the natural interpolation formula used in the quadrature method and the convergence of this method are also mentioned. A direct iterative method for the numerical solution of singular integral equations is also described. Finally, fifteen mathematical problems concerning possible generalizations of the above methods are also reported. An extensive bibliography is also included in this technical report.

Keywords Cauchy-type singular integral equations · Direct methods · Numerical solution · Galerkin method · Collocation method · Quadrature–collocation method · Quadrature method · Numerical integration rules · Quadrature rules · Cauchy-type integrals · Principal value integrals · Natural interpolation formula · Convergence · Iterative methods

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1. Introduction

Singular integral equations with Cauchy-type kernels on a finite interval, which without loss of generality is assumed coinciding with the interval \([-1, 1]\), appear in a large number of problems of physics and engineering. Here we will briefly describe the most known methods of numerical solution of such equations, which, in the sequel, for convenience are simply called singular integral equations. Such methods were mainly developed during the last fifteen years with a great delay in comparison with the corresponding methods for Fredholm integral equations; see, e.g., the paper of Nyström [1] for Fredholm integral equations published in 1930.

Now we consider the singular integral equation of the second kind

\[
a(x) \phi(x) + \int_{-1}^{1} K(t, x) \phi(t) \, dt = f(x), \quad -1 < x < 1.
\]

In this equation, the coefficient \(a(x)\), the kernel \(K(t, x)\) and the right-hand side function \(f(x)\) are known functions whereas the function \(\phi(x)\) is the unknown function, which has to be determined. The kernel \(K(t, x)\) of Eq. (1) presents a Cauchy-type singularity at the point \(t = x\) having the form

\[
K(t, x) = \frac{b(x)}{\pi} \frac{1}{t - x} + k(t, x),
\]

where \(b(x)\) and \(k(t, x)\) are known regular functions. In this way, Eq. (1) can be written as

\[
a(x) \phi(x) + \frac{b(x)}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t - x} \, dt + \int_{-1}^{1} k(t, x) \phi(t) \, dt = f(x), \quad -1 < x < 1.
\]

At this point, we note that a Cauchy-type integral in the principal value sense is defined as

\[
\int_{-1}^{1} \frac{\phi(t)}{t - x} \, dt := \lim_{\varepsilon \to 0^+} \left[ \int_{-1}^{x - \varepsilon} \frac{\phi(t)}{t - x} \, dt + \int_{x + \varepsilon}^{1} \frac{\phi(t)}{t - x} \, dt \right], \quad -1 < x < 1.
\]

Analogously to singular integral equations of the second kind of the form (1), we also have singular integral equations of the first kind of the somewhat simpler form

\[
\int_{-1}^{1} K(t, x) \phi(t) \, dt = f(x), \quad -1 < x < 1,
\]

where the kernel \(K(t, x)\) is again determined by Eq. (2). The simplest singular integral equation of the first kind results in if \(K(t, x) = 1/|\pi(t - x)|\) and it has the form

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t - x} \, dt = f(x), \quad -1 < x < 1.
\]

Although this singular integral equation possesses a closed-form solution, it can also be solved by expanding the unknown function \(\phi(t)\) into a series. This method was mentioned by Collatz [2] in 1955. To this end, we can use the known formula

\[
\frac{1}{\pi} \int_{0}^{\pi} \frac{\cos n \psi}{\cos \psi - \cos \omega} \, d\psi = \frac{\sin n \omega}{\sin \omega}
\]
by using the new variables $\psi$ and $\omega$. These variables are related to the original variables $t$ and $x$ as follows:

$$t = -\cos \psi \quad \text{and} \quad x = -\cos \omega.$$  \hspace{1cm} (8)

Now, in Eq. (5), as new unknown function we consider the function

$$g(\psi) = \phi(-\cos \psi) \sin \psi = \sum_{n=0}^{\infty} b_n \cos n \psi.$$  \hspace{1cm} (9)

(As usual, the prime in the above sum denotes that its first term $b_0$ should be halved.) Here because of Eq. (7), the coefficients $b_n$ coincide with the corresponding coefficients of the expansion of $f(x)$ into an analogous but now sine series, i.e.

$$f(-\cos \omega) \sin \omega = -\sum_{n=1}^{\infty} b_n \sin n \omega.$$  \hspace{1cm} (10)

Because of Eq. (9), for the unknown function $\phi(t)$ in Eq. (6) we have

$$\phi(t) = \frac{1}{\sqrt{1-t^2}} \sum_{n=0}^{\infty} b_n \cos n \psi = \sum_{n=0}^{\infty} b_n \frac{\cos n \psi}{\sin \psi}.$$  \hspace{1cm} (11)

In this equation, the coefficient $b_0$ cannot be determined and it remains arbitrary. But usually a supplementary condition of the form

$$\int_{-1}^{1} \phi(t) \, dt = C$$  \hspace{1cm} (12)

(with $C$ denoting a known constant frequently equal to zero) accompanies Eq. (6) with the coefficient $b_0$ now determined by using the above condition (12). Therefore, by taking into account the condition (12), the singular integral equation (6) has only one solution.

A generalization of the above method, which is appropriate for singular integral equations of the first kind of the form

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t-x} \, dt + \int_{-1}^{1} k(t,x) \phi(t) \, dt = f(x), \quad -1 < x < 1,$$  \hspace{1cm} (13)

was proposed by Kalandiya [3, 4] in 1959. Kalandiya’s method can be called quadrature–collocation method.

Singular integral equations of the second kind with constant coefficients of the form

$$a \phi(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t-x} \, dt + \int_{-1}^{1} k(t,x) \phi(t) \, dt = f(x), \quad -1 < x < 1,$$  \hspace{1cm} (14)

where $a$ and $b$ are now assumed to be constants, are, obviously, more general than Eq. (13). For this type of singular integral equations it can theoretically be proved that the unknown function $\phi(x)$ presents singularities at the ends $x = \pm 1$ of the integration interval $[-1,1]$ having the form

$$\phi(x) = w(x) g(x).$$  \hspace{1cm} (15)

Here $g(x)$ is a regular function and $w(x)$ is a weight function of the form

$$w(x) = (1-x)^\gamma (1+x)^\delta$$  \hspace{1cm} (16)

with the constants $\gamma$ and $\delta$ determined by the formulae

$$-\cot \pi \gamma = \cot \pi \delta = \frac{a}{b}, \quad -1 < \gamma, \delta < 0.$$  \hspace{1cm} (17)
Now we assume the new unknown function $g(x)$, which is defined by Eq. (15), to be approximated by a function $g_n(x)$ which has the form of a finite series of Jacobi polynomials $P^{(\gamma, \delta)}_i(x)$, i.e.

$$g(x) \approx g_n(x) = \sum_{i=0}^{n} c_i P^{(\gamma, \delta)}_i(x),$$  \hspace{1cm} (18)

where $c_i$ are unknown coefficients to be determined. Next, we take into account the known formula

$$aw(x)P^{(\gamma, \delta)}_n(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{P^{(\gamma, \delta)}_n(t)}{t-x} \, dt = -\frac{b}{2\sin \gamma} P^{(\gamma, \delta)}_{n-1}(x).$$  \hspace{1cm} (19)

Obviously, this formula is more general than the corresponding formula

$$\frac{1}{\pi} \int_{-1}^{1} w(t) \frac{T_n(t)}{t-x} \, dt = U_{n-1}(x) \quad \text{when} \quad \gamma = \delta = -\frac{1}{2},$$  \hspace{1cm} (20)

which is valid for the Chebyshev polynomials of the first kind $T_n(x)$ and the second kind $U_{n-1}(x)$.

Now, by using Eq. (19), we can determine the unknown coefficients $c_i$ in Eq. (18) in various ways. To this end, we demand that $g_n(x)$ satisfy the singular integral equation (14). Because of Eq. (15), Eq. (14) can also be written in its final form

$$aw(x)g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt + \int_{-1}^{1} w(t)k(t,x)g(t) \, dt = f(x), \quad -1 < x < 1.$$  \hspace{1cm} (21)

In general, this equation does not possess a unique solution. Therefore, a supplementary condition of the form (12) and here

$$\int_{-1}^{1} w(t)g(t) \, dt = C,$$  \hspace{1cm} (22)

where $C$ denotes a known constant, should also be satisfied by the unknown function $g(x)$ in the singular integral equation (21) in order that this equation can have a unique solution.

Before we proceed to a review of the methods of numerical solution of the singular integral equation (21) supplemented by the condition (22), we can mention that significant results in the development of such methods were obtained by the following researchers/research groups:

- By Erdogan and his collaborators at Lehigh University (1968–1978) [5–9].
- By Tsamasphyros and Theocaris and their collaborators at the National Technical University of Athens (1976–today) [15–20].
- By Ioakimidis, Theocaris and their collaborators also at the National Technical University of Athens (1973–1980) [21–45].
- By Ioakimidis at the University of Patras (1980–today) [46–69].
- By Elliott and his collaborators at the University of Tasmania (1977–today) [70–79].
- By Golberg and his collaborators at the University of Nevada (1978–today) [80–85].
- By Gerasoulis at Rutgers University and Srivastav and his collaborators at the State University of New York at Stony Brook (1979–today) [86–97].

Here we simply present a review of the fundamental results concerning the already existing methods of numerical solution of singular integral equations and described in the above references.


Finally, we can mention that the results concerning (i) the theory of singular integral equations [117, 118], (ii) the methods of numerical solution of Fredholm integral equations of the second kind [119, 120], (iii) the methods of numerical integration [121] and (iv) the theory orthogonal polynomials [122] are related and, therefore, of sufficient interest to the results reviewed in the present technical report.

2. The Galerkin method

In this section, we will apply the Galerkin method to the direct approximate solution of the singular integral equation (21), i.e. the equation

\[
aw(x)g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt + \int_{-1}^{1} w(t)k(t,x)g(t) \, dt = f(x), \quad -1 < x < 1, \tag{23}
\]

together with the supplementary condition (22) here with \( C = 0 \), i.e.

\[
\int_{-1}^{1} w(t)g(t) \, dt = 0. \tag{24}
\]

For the approximate solution of Eq. (23) by the Galerkin method, because of the form (16) of the weight function \( w(x) \), we use the Jacobi polynomials \( P_i^{(\gamma, \delta)}(x) \) of degree \( i \) corresponding to the orders of singularity \( \gamma \) and \( \delta \) of the weight function \( w(x) \) in Eq. (16). Moreover, for convenience in notation, we define the polynomials \( \phi_i(x) \) and \( \phi_i^*(x) \) simply as

\[
\phi_i(x) := P_i^{(\gamma, \delta)}(x) \quad \text{and} \quad \phi_i^*(x) := P_i^{(-\gamma,-\delta)}(x). \tag{25}
\]

Here we seek for an approximation \( g_n(x) \) to the unknown function \( g(x) \) in Eqs. (23) and (24) under the form of a sum of Jacobi polynomials, i.e.

\[
g(x) \approx g_n(x) = \sum_{i=0}^{n} c_i \phi_i(x), \tag{26}
\]

where \( c_i \) are unknown coefficients to be determined. Analogously, the two known functions \( k(t,x) \) and \( f(x) \) in Eq. (23) can be approximated by new functions \( k_n(t,x) \) and \( f_n(x) \), respectively, expressed, analogously, in the form of sums of Jacobi polynomials of the variable \( x \) as follows:

\[
k_n(t,x) = \sum_{k=0}^{n-1} \delta_k(t) \phi_k^*(x) \quad \text{and} \quad f_n(x) = \sum_{k=0}^{n-1} \varepsilon_k \phi_k^*(x). \tag{27}
\]

Here the coefficients \( \delta_k(t) \) and \( \varepsilon_k \) can be determined by using the known functions \( k(t,x) \) and \( f(x) \), respectively, together with the related integral formulæ.

The approximation of \( g(x) \) by \( g_n(x) \) substituted in the singular integral equation (23) gives

\[
\sum_{i=0}^{n} c_i \left[ -2^{-\kappa} b \ \cosec(\pi \gamma) \phi_i^* - h_i(x) \right] - f(x) = \eta^*(x), \quad -1 < x < 1. \tag{28}
\]

Here \( \kappa \) denotes the index of the singular integral equation (23), \( \eta^*(x) \) is the function denoting the error term and the new functions \( h_i(x) \) are defined as the inner products

\[
h_i(x) = (\phi_i, k) \tag{29}
\]
with the following definitions of the inner products of two functions

\[ (f, g) := \int_{-1}^{1} w(t)f(t)g(t) \, dt, \]  

\[ (f, g)^* := \int_{-1}^{1} w^*(t)f(t)g(t) \, dt. \]  

In the above Eq. (31), the new weight function \( w^*(t) \) is defined as

\[ w^*(t) := (1-t)^{-\gamma}(1+t)^{-\delta} = \frac{1}{w(t)}. \]  

During the application of the Galerkin method to the direct numerical solution of Eq. (23) we ask that

\[ (\eta^*, \phi_k^*)^* = 0, \quad k = 0, 1, \ldots, n - \kappa. \]  

Then we get the following system of linear algebraic equations with respect to the unknown coefficients \( c_i \) in the expansion (26) of \( g_n(x) \):

\[ -2^{-\kappa}b\theta_k^* \csc(\pi\gamma)c_{k+\kappa} + \sum_{i=0}^{n} d_{ik}c_i = F_k, \quad k = 0, 1, \ldots, n - \kappa. \]  

Here

\[ d_{ik} := (\phi_i^*, h_i)^* \quad \text{and} \quad F_k := (\phi_k^*, f)^*, \]  

and, moreover, we have also taken into account the orthogonality properties of the Jacobi polynomials

\[ (\phi_i^*, \phi_k^*)^* = 0, \quad i \neq k \quad \text{and} \quad (\phi_i^*, \phi_k^*)^* = \theta_k^*, \quad i = k, \]  

where the symbols \( \theta_k^* \) denote appropriate constants.

As far as the convergence of \( g_n(x) \) to \( g(x) \) as \( n \to \infty \) is concerned, in the case of singular integral equations of the first kind (with \( a = 0 \)) and with the known functions \( f \) and \( k \) possessing continuous derivatives of order \( p+1 \) with respect to the variable \( x \) on the interval \([-1, 1]\), i.e. \( f \in C^{p+1}[-1, 1] \) and \( k \in C^{p+1}[-1, 1] \), the following result holds true:

\[ \|g - g_n\|_\infty = O(n^{-p}). \]  

Here as usual the maximum norm \( \|g\|_\infty \) of a continuous function \( g(x) \) is defined by

\[ \|g\|_\infty = \max_{-1 \leq x \leq 1} |g(x)|. \]  

For the investigation of the convergence of \( g_n(x) \) to \( g(x) \) in the general case of singular integral equations of the form (23) of the first or the second kind, we consider the equivalent Fredholm integral equation of the second kind

\[ g(x) + \int_{-1}^{1} K(x,y)g(y) \, dy = F(x) \]  

with kernel

\[ K(x,y) = \frac{w(y)}{a^2 + b^2} \left[ aw^*(x)k(y,x) - b \frac{1}{\pi} \int_{-1}^{1} w^*(t) \frac{k(y,t)}{t-x} \, dt \right] \]  

and right-hand side function

\[ F(x) = \frac{1}{a^2 + b^2} \left[ aw^*(x)f(x) - b \frac{1}{\pi} \int_{-1}^{1} w^*(t) \frac{f(t)}{t-x} \, dt \right]. \]
Under these conditions the Galerkin method under consideration for the approximate solution of the singular integral equation (23) is equivalent to the solution of the following approximation of the Fredholm integral equation of the second kind (39):

$$g_n(x) + \int_{-1}^{1} K_n(x,y) g_n(y) \, dy = F_n(x)$$

with the new functions $K_n(x,y)$ and $F_n(x)$ given by equations analogous to Eqs. (40) and (41), respectively, but with the functions $k(t,x)$ and $f(x)$ there substituted by their approximations $k_n(t,x)$ and $f_n(x)$, respectively, already defined by Eqs. (27).

Under these conditions it results that if the $p_1$-derivative of the function $f(x)$ and the $p_2$-derivative of the function $k(t,x)$ with respect to the variable $x$ are Hölder-continuous functions and, simultaneously, the function $k(t,x)$ is continuous with respect to the variable $t$, we have

$$\lim_{n \to \infty} ||k - k_n||_\infty = 0 \quad \text{and} \quad \lim_{n \to \infty} ||f - f_n||_\infty = 0$$

and, next,

$$\lim_{n \to \infty} ||\mathcal{K} - \mathcal{K}_n||_\infty = 0 \quad \text{and} \quad \lim_{n \to \infty} ||F - F_n||_\infty = 0$$

with the following definitions of the functions $\mathcal{K}(x)$ and $\mathcal{K}_n(x)$:

$$\mathcal{K}(x) := \int_{-1}^{1} |K(x,y)| \, dy \quad \text{and} \quad \mathcal{K}_n(x) := \int_{-1}^{1} |K_n(x,y)| \, dy$$

Under these conditions it results that $g_n(x)$ converges to $g(x)$ with respect to the maximum norm, i.e.

$$\lim_{n \to \infty} ||g - g_n||_\infty = 0$$

and, in a more clear way,

$$||g - g_n||_\infty = O(n^{-\gamma^*})$$

for sufficiently large values of $n$ and with $\gamma^*$ an appropriate constant equal about to min($p_1, p_2$).

Alternatively, we can obtain similar results if instead of the Fredholm integral equation of the second kind (39) we consider the Fredholm integral equation of the second kind

$$G(x) + \int_{-1}^{1} K^*(x,y) G(y) \, dy = f(x),$$

which directly results from the singular integral equation (23) after the definition of the new unknown function $G(x)$ as

$$G(x) := aw(x) g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt.$$

Exactly as previously, if Eqs. (44) hold true, then by the use of the Galerkin method the convergence of the approximation $G_n(x)$ to $G(x)$ results in as $n \to \infty$ and, next, the convergence of $g_n(x)$ to $g(x)$.

3. The collocation method

For the numerical solution of singular integral equations of the form (23) together with the supplementary condition (24) analogous to the Galerkin method is the collocation method. Again in this method, approximations of the forms (27) are used for the known functions $k(t,x)$ and $f(x)$ in Eq. (23) but now the functions $\delta_k(t)$ and the constants $\varepsilon_k$ in Eqs. (27) are chosen in such a way that

$$k_n(t,x_k) = k(t,x_k) \quad \text{and} \quad f_n(x_k) = f(x_k), \quad k = 0, 1, \ldots, n-1.$$
Here the collocation points $x_k$ are determined by
\[ \phi_n^*(x_k) = 0, \quad k = 0, 1, \ldots, n - 1. \] (51)

4. Numerical integration rules

Another method for the numerical solution of singular integral equations is based on the use of numerical integration rules (or quadrature rules) for the approximation of the related integrals. In this section, we study these numerical integration rules.

At first, for ordinary integrals we can use interpolatory quadrature rules of the form
\[ \int_{-1}^{1} w(t) g(t) \, dt \approx \sum_{i=1}^{n} A_i g(t_i), \quad t_i \in [-1, 1], \quad i = 1, 2, \ldots, n, \] (52)
where $w(t)$ is the weight function, $t_i$ are the nodes and $A_i$ are the weights. Clearly, both the nodes $t_i$ and the weights $A_i$ depend on the number of nodes $n$.

Now, in the case of Cauchy-type principal value integrals, we can simply write
\[ \int_{-1}^{1} \frac{g(t)}{t-x} \, dt = \int_{-1}^{1} w(t) \frac{g(t) - g(x)}{t-x} \, dt + g(x) \int_{-1}^{1} w(t) \, dt \] (53)
(subtraction of the singularity). Next, on the basis of the quadrature rule (52) for ordinary integrals, we approximate the first integral in the right-hand side of Eq. (53). Then from this equation we get
\[ \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt \approx \sum_{i=1}^{n} A_i \frac{g(t_i) - g(x)}{t_i-x} + q_0(x) g(x), \quad x \neq t_i \quad i = 1, 2, \ldots, n, \quad g \in C^1[-1, 1]. \] (54)

In this section, we use the functions
\[ q_n(x) := \int_{-1}^{1} w(t) \frac{\sigma_n(t)}{t-x} \, dt, \quad -1 < x < 1, \quad n = 0, 1, \ldots, \] (55)
the first of which, $q_0(x)$, is essentially already present in the last term of Eq. (53). In the above definition (55) of the functions $q_n(x)$, the functions $\sigma_n(x)$ simply denote the polynomials
\[ \sigma_n(x) := \prod_{i=1}^{n} (x-t_i). \] (56)

Then the quadrature rule (54) for Cauchy-type principal value integrals takes its final form
\[ \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt \approx \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-x} + q_n(x) g(x), \quad x \neq t_i \quad i = 1, 2, \ldots, n. \] (57)
Now for $g(x) = \sigma_n(x)$ and, therefore, $g_n(t_i) = \sigma_n(t_i) = 0$ because of Eq. (56), we easily find from Eq. (54) that
\[ q_0(x) = \sum_{i=1}^{n} A_i \frac{1}{t_i-x} + \frac{q_n(x)}{\sigma_n(x)}, \quad x \neq t_i \quad i = 1, 2, \ldots, n. \] (58)

Then the quadrature rule (57) can also be written in the somewhat simpler form
\[ \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt \approx \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-x} + M_n(x) g(x), \quad x \neq t_i \quad i = 1, 2, \ldots, n, \] (59)
where
\[ M_n(x) := \frac{q_n(x)}{\sigma_n(x)}. \] (60)
In the case where the variable \( x \) coincides with one of the nodes \( t_m \), the following quadrature rule results in:

\[
\int_{-1}^{1} w(t) \frac{g(t)}{t-t_m} \, dt \approx \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-t_m} + A_m g'(t_m) + A_{mn}g(t_m), \quad m = 1, 2, \ldots, n, \tag{61}
\]

where the constants \( A_{mn} \) are determined by

\[
A_{mn} = \frac{1}{\sigma_n'(t_m)} \left[ q_n'(t_m) - \frac{1}{2} A_m \sigma_n''(t_m) \right]. \tag{62}
\]

The above computation of a Cauchy-type principal value integral has been based on its reduction to an ordinary integral on the basis of Eq. (53) (subtraction of the singularity). Thinking in a different way, we can alternatively use the Cauchy residue theorem for the integral

\[
I_0 = \frac{1}{2\pi i} \oint_C \frac{g(\zeta)}{(\zeta-x)(\zeta-z)\sigma_n(\zeta)} \, d\zeta, \tag{63}
\]

where \( C \) is a smooth closed contour surrounding the integration interval \([-1, 1]\). Then we get the formula

\[
\frac{g(z)}{(z-x)\sigma_n(z)} = \sum_{i=1}^{n} \frac{g(t_i)}{t_i-x} + \frac{1}{(z-x)\sigma_n(x)}. \tag{64}
\]

From this formula the quadrature rule (59) for Cauchy-type principal value integrals results in again.

In a completely different way of thinking, we can alternatively use the Plemelj formulae for the generally complex functions \( q_n(z) \) defined by Eq. (55) for \( z \in (-1, 1) \) (as real functions) and by the analogous equation

\[
q_n(z) := \int_{-1}^{1} w(t) \frac{\sigma_n(t)}{t-z} \, dt \tag{65}
\]

for \( z \not\in [-1, 1] \) (as complex functions). Then, on the basis of these formulae, we get

\[
q_n^+(x) - q_n^-(x) = 2\pi i w(x) \sigma_n(x), \quad x \in (-1, 1), \tag{66}
\]

\[
q_n^+(x) + q_n^-(x) = 2 \int_{-1}^{1} w(t) \frac{\sigma_n(t)}{t-x} \, dt \quad x \in (-1, 1). \tag{67}
\]

Analogously, we can define the generally complex function \( \Phi(z) \) as

\[
\Phi(z) := \int_{-1}^{1} w(t) \frac{g(t)}{t-z} \, dt, \quad z \not\in [-1, 1], \tag{68}
\]

\[
\Phi(x) := \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt, \quad x \in (-1, 1). \tag{69}
\]

Then the second Plemelj formula (67) takes the form

\[
\Phi(x) = \frac{1}{2} \left[ \Phi^+(x) + \Phi^-(x) \right]. \tag{70}
\]

On the basis of this formula, a Cauchy-type principal value integral can be computed as the mean value of two Cauchy-type integrals but not in the principal-value sense. By using Eq. (67), the quadrature rule (59) for Cauchy-type principal value integrals results in again.
A final approach to the derivation of the quadrature rule (59) for Cauchy-type integrals simply consists in the use of the definition of a Cauchy-type principal value integral. Hence, we have

\[ \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt = \lim_{\varepsilon \to 0} \left[ \int_{-1}^{x-\varepsilon} w(t) \frac{g(t)}{t-x} \, dt + \int_{x+\varepsilon}^{1} w(t) \frac{g(t)}{t-x} \, dt \right]. \tag{71} \]

In this way, the computation of a principal value integral is simply reduced to the computation of two ordinary integrals on the intervals \([-1, x-\varepsilon]\) and \([x+\varepsilon, 1]\).

5. The quadrature–collocation method

According to the quadrature–collocation method, we use the two quadrature rules

\[ \int_{-1}^{1} w(t) g(t) \, dt \approx \sum_{i=1}^{n} A_i g(t_i), \quad \phi_n(t_i) = 0, \quad i = 1, 2, \ldots, n, \tag{72} \]

\[ \int_{-1}^{1} w^*(t) g(t) \, dt \approx \sum_{k=1}^{n} B_k g(x_k), \quad \phi^*_{n-1}(x_k) = 0, \quad k = 1, 2, \ldots, n-1, \tag{73} \]

with the two weight functions \(w(t)\) and \(w^*(t)\) in these quadrature rules defined by Eqs. (16) and (32), respectively. By using the quadrature–collocation method, we approximate the singular integral equation (23) together with the supplementary condition (24) or better the equivalent Fredholm integral equation of the second kind (39) essentially by the Fredholm integral equation (42) (because of numerical integrations according to the developments of the previous section), but now with

\[ K_n(x,y) = \sum_{i=0}^{n-2} \sum_{k=0}^{n-1} \beta_{ik} \phi_k^*(x) \phi_i(y), \tag{74} \]

\[ F_n(x) = \sum_{k=0}^{n-2} \varepsilon_k \phi_k^*(x), \tag{75} \]

where \(\beta_{ik}\) and \(\varepsilon_k\) denote appropriate coefficients.

Then for the unknown function \(g(x)\) in Eq. (23) we get an approximation \(g_n(x)\) of the form

\[ g_n(x) = \sum_{i=0}^{n-1} \delta_i \phi_i(x). \tag{76} \]

In complete analogy to the developments of Section 2 for the Galerkin method, for the convergence of the present quadrature–collocation method it can be proved that the convergence-related Eq. (47) is valid again, but now with respect to the present quadrature–collocation method exactly as has been already the case with respect to the Galerkin method.

6. The quadrature method and the natural interpolation formula

In the quadrature method, at first, let us consider an ordinary Fredholm integral equation of the second kind of the form

\[ \phi(x) + \int_{-1}^{1} K(t,x) \phi(t) \, dt = f(x), \quad -1 \leq x \leq 1, \tag{77} \]

where \(\phi(x)\) is the unknown function, \(K(t,x)\) is the kernel and \(f(x)\) is the right-hand side function. For the approximate solution of Eq. (77) we can apply a numerical integration rule (or, equivalently, a quadrature rule) of the form (72). Then we get the approximation

\[ \phi_n(x) + \sum_{i=1}^{n} A_i K(t_i,x) \phi_n(t_i) = f(x), \quad -1 \leq x \leq 1, \tag{78} \]
In this way, we directly obtain the following system of linear algebraic equations:

\[
\phi_n(t_k) + \sum_{i=1}^{n} A_i K(t_i, t_k) \phi_n(t_i) = f(t_k), \quad k = 1, 2, \ldots, n. \tag{79}
\]

The values of \( \phi_n(t_i) \) result from the solution of the above system. Next, Eq. (78) can be used as a natural interpolation formula for the determination of the approximation \( \phi_n(x) \) to the unknown function \( \phi(x) \) of the original Fredholm integral equation of the second kind (77) on the whole integration interval \([-1, 1]\). Hence, we have

\[
\phi_n(x) = f(x) - \sum_{i=1}^{n} A_i K(t_i, x) \phi_n(t_i). \tag{80}
\]

The above results for Fredholm integral equations of the second kind can be generalized to singular integral equations of the form (3). Because of Eq. (15) for the weight function \( w(x) \), the singular integral equation (3) can also be written as

\[
a(x)w(x)g(x) + \frac{b(x)}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt + \int_{-1}^{1} w(t) k(t, x) g(t) \, dt = f(x), \quad -1 < x < 1, \tag{81}
\]

where \( a(x), b(x), k(t, x) \) and \( f(x) \) are known functions, \( g(x) \) is the unknown function and \( w(x) \) is the weight function. In general, such an equation is accompanied by a condition of one of the forms

\[
\int_{-1}^{1} w(t) g(t) \, dt = C \quad \text{or} \quad g(-1) = 0 \quad \text{or} \quad g(1) = 0, \tag{82}
\]

where \( C \) is a known constant and frequently \( C = 0 \). In this way, Eq. (81) will have a unique solution.

By applying the aforementioned quadrature rules (52) for ordinary integrals and (59) for Cauchy-type principal value integrals to the singular integral equation (81), we get

\[
\sum_{i=1}^{n} A_i \left[ \frac{b(x)}{\pi(t_i-x)} + k(t_i, x) \right] g_n(t_i) + N_n(x) g_n(x) = f(x), \quad x \neq t_i, \quad i = 1, 2, \ldots, n. \tag{83}
\]

where \( N_n(x) \) is an easily determined function. Next, we apply Eq. (83) to the roots \( x_k \) of the function \( N_n(x) \) (the collocation points), i.e.

\[
N_n(x_k) = 0, \quad k = 1, 2, \ldots, n-1. \tag{84}
\]

In this way, we directly obtain the following system of linear algebraic equations:

\[
\sum_{i=1}^{n} A_i \left[ \frac{b(x_k)}{\pi(t_i-x_k)} + k(t_i, x_k) \right] g_n(t_i) = f(x_k), \quad k = 1, 2, \ldots, n-1. \tag{85}
\]

This system is supplemented by one more linear algebraic equation resulting from one of the three conditions (82), i.e. (after numerical integration to the integral of the first of them)

\[
\sum_{i=1}^{n} A_i g_n(t_i) = C \quad \text{or} \quad g_n(-1) = 0 \quad \text{or} \quad g_n(1) = 0 \tag{86}
\]

with the function \( g_n(x) \) constituting again an approximation to the really unknown function \( g(x) \).

After the solution of the system of linear algebraic equations (85) (together with one of Eqs. (86)) and the determination of the values of \( g_n(t_i) \), Eq. (83) can be used as a natural interpolation formula.
analogously to the Nyström natural interpolation formula (80) for Fredholm integral equations of the second kind. Hence, from Eq. (83) we finally get

\[ g_n(x) = \frac{1}{N_n(x)} \left\{ f(x) - \sum_{i=1}^{n} A_i \left[ \frac{b(x)}{\pi (t_i - x)} + k(t_i, x) \right] g_n(t_i) \right\}, \]

\[ x \neq t_i, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad x \neq x_k, \quad k = 1, 2, \ldots, n - 1, \tag{87} \]

since \( N_n(x_k) = 0 \), Eq. (84). Moreover, at the nodes \( t_i \) we obviously have

\[ g_n(x) = g_n(t_i), \quad x = t_i, \quad i = 1, 2, \ldots, n. \tag{88} \]

On the contrary, for the points \( x_k \) of application of Eq. (83), i.e. for the collocation points, a limiting procedure should be applied to Eq. (87) so that the corresponding approximate values \( g_n(x_k) \) of the unknown function \( g(x) \) at the collocation points \( x_k \) can be determined.

7. Convergence of the quadrature method

In this section, we will mention some results concerning the convergence of the quadrature method already described in the previous section and making use of the corresponding natural interpolation formula (87). At first, we assume that the two functions \( a(x) \) and \( b(x) \) in Eq. (81) (the coefficients of this equation) are simply constants and that it is the first of conditions (82) with \( C = 0 \) that is valid here. Therefore, we have

\[ aw(x)g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt + \int_{-1}^{1} w(t)k(t,x)g(t) \, dt = f(x), \quad -1 < x < 1, \tag{89} \]

and

\[ \int_{-1}^{1} w(t)g(t) \, dt = 0. \tag{90} \]

Now by using the quadrature rule (59) for Cauchy-type principal value integrals, we find that

\[ aw(x)g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt \approx \frac{b}{\pi} \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-x} + N_n(x)g(x), \quad -1 < x < 1. \tag{91} \]

We also use the quadrature rule (52) for ordinary integrals, i.e.

\[ \int_{-1}^{1} w(t)g(t) \, dt \approx \sum_{i=1}^{n} A_i g(t_i). \tag{92} \]

Now we can apply both quadrature rules (91) and (92) to the singular integral equation (89) and the quadrature rule (92) to the accompanying condition (90). In this way, we find that

\[ \sum_{i=1}^{n} A_i \left[ \frac{b}{\pi (t_i-x_k)} + k(t_i, x_k) \right] g_n(t_i) = f(x_k), \quad k = 1, 2, \ldots, n - 1, \tag{93} \]

and

\[ \sum_{i=1}^{n} A_i g_n(t_i) = 0. \tag{94} \]

In the above Eqs. (93) the roots \( x_k \) of the function \( N_n(x) \) appearing in the quadrature rule (91) were used again as points of application of Eq. (89) (collocation points). After the numerical solution of the system of linear algebraic equations (93) and (94) the function \( g_n(x) \) can be defined on the whole integration interval \([-1, 1]\] by using the natural interpolation formula (87) in accordance with the developments of the previous section.
Next, for the study of the convergence of the above method we define the new function

\[ G(x) := aw(x)g(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt, \quad -1 < x < 1. \]  

(95)

Then the singular integral equation (89) takes the simpler, modified form

\[ G(x) = f(x) - \int_{-1}^{1} w(t)k(t,x)g(t) \, dt. \]  

(96)

In a similar way, we define the new function

\[ G_n(x) := aw(x)g_n(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g_n(t)}{t-x} \, dt, \quad -1 < x < 1, \]  

(97)

which can be considered as an approximation to \( G(x) \) in Eq. (95). Because of the definition of \( g_n(x) \) and the original singular integral equation (89), we further find that

\[ G_n(x) = f(x) - \sum_{i=1}^{n} A_i k(t_i,x)g_n(t_i). \]  

(98)

Next, taking into account the natural interpolation formula (87) as well as the definitions (95) and (97) of the two functions \( G(x) \) and \( G_n(x) \) respectively, we directly find that

\[ g(x) = \frac{1}{a^2 + b^2} \left[ a w^*(x) G(x) - \frac{b}{\pi} \int_{-1}^{1} w^*(t) \frac{G(t)}{t-x} \, dt \right], \]  

(99)

\[ g_n(x) = \frac{1}{a^2 + b^2} \left[ a w^*(x) G_n(x) - \frac{b}{\pi} \int_{-1}^{1} w^*(t) \frac{G_n(t)}{t-x} \, dt \right]. \]  

(100)

For the proof of the convergence of \( g_n(x) \) to \( g(x) \) for increasing values of \( n \), i.e.

\[ ||g - g_n||_\infty \to 0 \quad \text{as} \quad n \to \infty \]  

(101)

we use the above convergence result only at the nodes \( t_i \), i.e.

\[ \max_{i=1,2,...,n} |g(t_i) - g_n(t_i)| \to 0 \quad \text{as} \quad n \to \infty. \]  

(102)

This limit holds true because of the convergence of the quadrature–collocation method. Next, at first we can prove the convergence of \( G_n(x) \) to \( G(x) \), i.e.

\[ ||G - G_n||_\infty \to 0 \quad \text{as} \quad n \to \infty \]  

(103)

on the basis of Eqs. (96) and (98). Finally, Eqs. (99) and (100) lead to the proof of the validity of the limit (101) concerning the convergence of \( g_n(x) \) to \( g(x) \).

8. Iterative methods

One of the first and classical methods of approximate solution of ordinary Fredholm integral equations of the second kind is the method of successive approximations. By adopting this method for the approximate solution of the Fredholm integral equation of the second kind of the form

\[ \phi(x) + \int_{-1}^{1} K(t,x)\phi(t) \, dt = f(x), \]  

(104)
where we can use one of the following two iterative algorithms:

\[ \phi_0(x) = f(x), \quad \phi_0^*(x) = f(x), \]  
\[ \phi_{n+1}(x) = f(x) - \int_{-1}^{1} K(t,x) \phi_n(t) \, dt, \quad n = 0, 1, \ldots, \]  
\[ \phi_{n+1}^*(x) = f(x) - \sum_{i=1}^{n} A_i K(t_i,x) \phi_n^*(t_i), \quad n = 0, 1, \ldots, \]  
where \( \phi_n(x) \) and \( \phi_n^*(x) \) denote two approximations of the exact solution \( \phi(x) \) of the Fredholm integral equation of the second kind (104). Evidently, the only difference between the iterative algorithms (106) and (107) consists in the use of a numerical integration rule in the iterative algorithm (107) for the approximation of the integral in the right-hand side of the iterative algorithm (106).

Now for Cauchy-type singular integral equations we can also use the above iterative algorithms but now after the conversion of such an integral equation (on the basis of a regularization method) to an equivalent Fredholm integral equation of the second kind.

It is also possible to directly use iterative methods for the approximate solution of Cauchy-type singular integral equations. Such a method is described below. This method concerns the singular integral equation

\[ \frac{1}{\pi} \int_{-1}^{1} w(t) \left[ \frac{1}{t-x} + k(t,x) \right] g(t) \, dt = f(x), \]  

where

\[ w(t) = \frac{1}{\sqrt{1-t^2}}, \]  

accompanied by the condition

\[ \int_{-1}^{1} w(t) g(t) \, dt = 0, \]  

which assures the uniqueness of the solution of the singular integral equation (108).

Now, we can use the roots of appropriate Chebyshev polynomials, i.e.

\[ t_i = \cos \left( \frac{(2i-1)\pi}{2n} \right), \quad i = 1, 2, \ldots, n, \]  
\[ u_j = \cos \frac{j\pi}{n}, \quad j = 0, 1, \ldots, n, \]  

and we introduce the kernel

\[ K(t,x) = \frac{1}{t-x} + k(t,x) \]  

and the notation

\[ \delta_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, n-1 \quad \text{and} \quad \delta_j = 1 \quad \text{for} \quad j = 0, n. \]  

Then we get the iterative algorithm defined by the following equations:

\[ g_{k+1}(0) = g_k(0) - \frac{1}{n} \sum_{i=1}^{n} g_k(t_i), \]  
\[ g_{k+1}(u_j) = (1-n\delta_j) g_k(u_j) + [1-u_j k(0,u_j)][g_{k+1}(0) - g_k(0)] \]  
\[ + u_j \left[ f(u_j) - \frac{1}{n} \sum_{i=1}^{n} K(t_i,u_j) g_k(t_i) \right], \quad j = 0, 1, \ldots, n, \]  
\[ g_{k+1}(t_i) = g_k(t_i) + [1-t_i k(0,t_i)][g_{k+1}(0) - g_k(0)] \]  
\[ + t_i \left[ f(t_i) - \frac{1}{n} \sum_{j=0}^{n}'' K(u_j,t_i) g_k(u_j) \right], \quad i = 1, 2, \ldots, n, \]
where the double prime in the sum of Eq. (117) denotes that its first and last terms should be halved.

Although the convergence of the above iterative algorithm has not been proved so far, nevertheless, from Eqs. (115) to (117) it is obvious that if this algorithm converges as \( k \to \infty \), then the following equations hold true:

\[
\sum_{i=1}^{n} g_{\infty}(t_i) = 0, \quad (118)
\]
\[
\frac{1}{n} \sum_{i=1}^{n} K(t_i, u_j) g_{\infty}(t_i) = f(u_j), \quad j = 0, 1, \ldots, n, \quad (119)
\]
\[
\frac{1}{n} \sum_{j=0}^{n} K(u_j, t_i) g_{\infty}(u_j) = f(t_i), \quad i = 1, 2, \ldots, n. \quad (120)
\]

It is directly observed that these equations simply constitute different expressions of Eqs. (110) and (108) but, evidently, after the use of numerical integration rules for the approximation to the integrals there. Surely, for large values of \( n \) (as \( n \to \infty \)) the above values \( g_{\infty}(t_i) \) tend to the exact values \( g(t_i) \) of the solution \( g(t) \) of Eqs. (108) and (110).

9. Various problems

Beyond the above results, there are several more mathematical problems related to Cauchy-type principal value integrals and the corresponding singular integral equations. Some of these problems are listed below:

1. Problems related to the numerical computation of two-dimensional Cauchy-type principal value integrals on a plane region \( S \) of the form

\[
I(x_0, y_0) = \iint_{S} f(\theta) \frac{u(x, y)}{(x-x_0)^2 + (y-y_0)^2} \, dx \, dy,
\]

where \( f(\theta) \) is the characteristic function and \( u(x, y) \) is the integrand, and further to the numerical solution of the related singular integral equations.

2. Problems related to the numerical solution of singular integral equations with generalized kernels of the form

\[
\frac{1}{\pi} \int_{0}^{1} w(t) \left[ \frac{1}{t-x} + \frac{\lambda}{t+x} + k(t, x) \right] g(t) \, dt = f(x).
\]

3. Generalizations of the results reviewed in the previous sections by using systems of orthogonal polynomials different from the Chebyshev and the Jacobi polynomials. Such non-classical polynomials appear, e.g., in singular integral equations with variable coefficients \( a(x) \) and \( b(x) \) of the general form

\[
a(x)w(x)g(x) + \frac{b(x)}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt + \int_{-1}^{1} w(t)k(t, x)g(t) \, dt = f(x).
\]

4. The investigation of the number of the collocation points \( x_k \) of the singular integral equations. For singular integral equations of the first kind, if a numerical integration rule of the form

\[
\int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt \approx \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-x} + \frac{q_n(x)}{\sigma_n(x)} g(x), \quad x \neq t_i, \quad i = 1, 2, \ldots, n,
\]
N. I. Ioakimidis: Methods of numerical solution of singular integral equations (1983)

is used, then the collocation points \( x_k \) are the roots of the function \( q_n(x) \). Now, applying the above numerical integration rule (124) with \( g(t) = 1 \), we directly find that

\[
q_0(x) = \sum_{i=1}^{n} \frac{A_i}{t_i - x} + \frac{q_n(x)}{\sigma_n(x)}, \quad x \neq t_i, \quad i = 1, 2, \ldots, n. \tag{125}
\]

From this numerical integration rule we directly conclude the existence of at least one root of the function \( q_n(x) \) in each open subinterval \((t_i, t_{i+1})\) of the integration interval \([-1, 1]\) with \( i = 1, 2, \ldots, n - 1 \). Therefore, there are at least \( n - 1 \) roots of the function \( q_n(x) \) in the integration interval \([-1, 1]\).

5. Generalizations of the results reviewed in the previous sections to the case of integrodifferential singular integral equations of the form

\[
a(x)\phi(x) + \frac{b(x)}{\pi} \int_{-1}^{1} \frac{\phi'(t)}{t-x} \, dt + \int_{-1}^{1} k(t,x)\phi'(t) \, dt = f(x),
\]

where again \( a(x), b(x), k(t,x) \) and \( f(x) \) are known functions and \( \phi(x) \) is the unknown function.

6. The completion of the proofs of equivalence of the direct methods of numerical solution of singular integral equations to the corresponding numerical methods for Fredholm integral equations of the second kind.

7. The completion of the proofs of convergence of the direct methods of numerical solution of singular integral equations.

8. The study of the possibility of an arbitrary selection of the collocation points \( x_k \) by using the Lagrange and Hermite interpolation formulae.

9. The study of singular integral equations with discontinuous kernels \( k(t,x) \) and/or discontinuous right-hand side functions \( f(x) \).

10. The study of the possibility of selection of the collocation points \( x_k \) outside the integration interval \([-1, 1]\). This case appears, e.g., when we use the Gauss–Jacobi numerical integration rule with a weight function \( w(x) \) of the form (16), but now with complex singularities \( \gamma \) and \( \delta \) at the ends of the integration interval \([-1, 1]\).

11. The study of singular integral equations with Hilbert-type kernels of the form

\[
a\phi(x) + \frac{b}{2\pi} \int_{0}^{2\pi} \cot \frac{t-x}{2} \phi(t) \, dt + \int_{0}^{2\pi} k(t,x)\phi(t) \, dt = f(x), \quad 0 \leq x \leq 2\pi. \tag{127}
\]

12. The study of singular integral equations with kernels presenting a logarithmic singularity of the form

\[
\int_{-1}^{1} w(t) [\log |t-x| + k(t,x)] g(t) \, dt = f(x) \tag{128}
\]

with a weight function \( w(t) \) again of the form (109). For the numerical solution of such a singular integral equation it is appropriate to differentiate it with respect to the free variable \( x \) and, in this way, to reduce it to a Cauchy-type singular integral equation of the form

\[
\int_{-1}^{1} w(t) \left[ -\frac{1}{t-x} + \frac{\partial k}{\partial x}(t,x) \right] g(t) \, dt = f'(x). \tag{129}
\]

Clearly, this equation can be solved by the methods already reviewed in the previous sections.
13. The numerical solution of singular integral equations on semi-infinite or infinite (instead of finite) integration intervals.

14. The numerical solution of singular integral equations on closed contours $C$ in the complex plane.

15. The modification of the theories of numerical solution and convergence of Fredholm integral equations of the second kind so that they can become applicable to singular integral equations.

Many of the above mathematical problems have already been studied (at least partially) in relatively recent papers. Nevertheless, much more should be done so that the numerical solution of singular integral equations appearing in practice ceases to be a problem (exactly as the numerical solution of the corresponding Fredholm integral equations has ceased to be a problem) and, simultaneously, a complete study of the conditions of convergence of the applied numerical methods for singular integral equations be available. In any case, it is worthy to note the emphasis already put on the numerical solution of singular integral equations during the last fifteen years. This emphasis guarantees the achievement of the above aims to a very satisfactory degree in the near future.

References


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2All the links (external links in blue) in this section were added by the author on 19 February 2018 for the online publication of this technical report. Moreover, final publication details were added in References [63–69].


