On the application of the generalized Plemelj formulas to crack problems

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Abstract The singular integral equation for the simple plane elasticity problem of a straight crack under a known normal pressure distribution is derived by using the first generalized Plemelj formula (where a finite-part integral appears) on the basis of the complex-variable formulation of plane elasticity problems. The present results are believed to prove useful in a lot of more complicated elasticity problems, to which they can easily be applied, and to lead to the wide use of singular integral equations with finite-part integrals in elasticity problems.

Keywords Crack problems · Fracture mechanics · Plane elasticity · Isotropic elasticity · Generalized Plemelj formulas · Principal value integrals · Cauchy-type integrals · Singular integral equations · Finite-part integrals · Hadamard-type integrals · Hypersingular integrals · Hypersingular integral equations · Analytic functions · Complex variables · Complex potentials

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1. The generalized Plemelj formulas

Consider the well-known Plemelj formulas \([1, 2]\) (also called the Sokhotski formulas in the recent Soviet literature)

\[
\phi^+(t) + \phi^-(t) = \frac{1}{\pi i} \int_L \frac{\mu(\tau)}{\tau - t} \, d\tau, \tag{1}
\]

\[
\phi^+(t) - \phi^-(t) = \mu(t), \tag{2}
\]

where \(L\) is a closed or open curve, \(\mu(t)\) a density function and \(\phi^\pm(t)\) the boundary values (along \(L\)) of the Cauchy-type integral

\[
\phi(z) = \frac{1}{2\pi i} \int_L \frac{\mu(\tau)}{\tau - z} \, d\tau, \quad z = x + iy \not\in L, \tag{3}
\]

which is an analytic function in the complex plane cut along \(L\). These formulas form the basis for the reduction of any problem of plane and antiplane elasticity (by the method of complex potentials) to a singular integral equation or to a system of such equations (see, e.g., References \([2, 3]\) and have been used hundreds of times in such problems. Clearly, the first Plemelj formula, Eq. (1), is the most important and it shows why principal value integrals appear in plane and antiplane elasticity problems.

Now we consider the \(n\)th derivative of the complex function \(\phi(z)\) in Eq. (3)

\[
\phi^{(n)}(z) = \frac{n!}{2\pi i} \int_L \frac{\mu(\tau)}{(\tau - z)^{n+1}} \, d\tau, \quad z = x + iy \not\in L. \tag{4}
\]

This function also possesses boundary values \(\phi^{(n)}(t)\) along \(L\) (with the exception of its end-points) and the following generalized Plemelj formulas \([4]\) hold true:

\[
\phi^{(n)+}(t) + \phi^{(n)-}(t) = \frac{n!}{\pi i} \int_L \frac{\mu(\tau)}{(\tau - t)^{n+1}} \, d\tau, \tag{5}
\]

\[
\phi^{(n)+}(t) - \phi^{(n)-}(t) = \mu^{(n)}(t), \tag{6}
\]

provided that \(\mu^{(n)}(t)\) exists and is Hölder-continuous. The case of noninteger values of \(n\) is also considered in Reference \([4]\). Moreover, the integral in the right-hand side of Eq. (5) should be interpreted as a finite-part integral of Hadamard \([5, 6]\). Quadrature rules for finite-part (or Hadamard-type) integrals have been suggested by Kutt \([6]\), Paget \([7]\) (with \(n = 1\) only) and Ioakimidis \([8, 9]\). Clearly, the first generalized Plemelj formula, Eq. (5), is again the most important and it shows why finite-part integrals can appear in plane and antiplane elasticity problems. Nevertheless, to this author’s best knowledge, this formula has never been used in this class of problems. An application of this formula to a simple crack problem, already having been reduced to a singular integral equation by other methods, is made in the next section.

2. Application to a crack problem

To illustrate the usefulness of the generalized Plemelj formulas (5) and (6) to the reduction of plane and antiplane, isotropic and anisotropic elasticity problems to singular integral equations, here we consider the problem of a straight crack \(L\) (or of an array of such cracks) along the \(Ox\)-axis in plane isotropic elasticity. The crack is assumed loaded by a known normal pressure distribution \(p(x)\) along both its edges. This classical problem can directly be reduced to the determination
of the complex potential \( \phi(z) \) of Kolosov–Muskhelishvili [2], in terms of which the stress components \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) can be determined as [10, 11]

\[
\begin{align*}
\sigma_x &= 2 \text{Re} \phi'(z) - 2y \text{Im} \phi''(z), \\
\sigma_y &= 2 \text{Re} \phi'(z) + 2y \text{Im} \phi''(z), \\
\tau_{xy} &= -2y \text{Re} \phi''(z).
\end{align*}
\]

The second of our two boundary conditions

\[
\sigma_y(x \pm 0i) = -p(x), \quad \tau_{xy}(x \pm 0i) = 0, \quad x \in L,
\]

is satisfied automatically. As regards the first one, because of Eq. (8) it is reduced to

\[
2 \text{Re} \phi'^\pm(x) = -p(x), \quad x \in L.
\]

Now we assume the complex potential \( \phi(z) \) to be of the form [11]

\[
\phi(z) = \int_L \frac{q(\tau)}{\tau - z} d\tau, \quad z \notin L,
\]

where \( q(\tau) \) is an unknown density function, and we differentiate this equation with respect to \( z \):

\[
\phi'(z) = \int_L \frac{q(\tau)}{(\tau - z)^2} d\tau, \quad z \notin L.
\]

Now, by using the generalized Plemelj formulas (5) and (6) with \( n = 1 \), we observe that

\[
\phi'^\pm(x) = \pm \pi i q'(x) + \int_L \frac{q(\tau)}{(\tau - x)^2} d\tau, \quad x \in L.
\]

It is further clear from Eqs. (11) and (14) that \( q(x) \) should be a real function and, afterwards, that it should satisfy the singular integral equation

\[
2 \int_L \frac{q(\tau)}{(\tau - x)^2} d\tau = -p(x), \quad x \in L.
\]

This equation is identical with the singular integral equation of Bueckner [11] for the same crack problem, which has the form

\[
2 \frac{d}{dx} \int_L \frac{q(\tau)}{\tau - x} d\tau = -p(x), \quad x \in L,
\]

as was observed in Reference [12], where Eq. (16) was directly transformed into Eq. (15) by performing the differentiation under the integral sign in Eq. (16). Here we derived Eq. (15) without our being based on Eq. (16), just by using the fundamental formulas (8) and (9) of the theory of elasticity.

We can add that (i) the unknown real density function \( q(\tau) \) in Eqs. (15) and (16) is proportional to the crack opening displacement [11] and, therefore, it should vanish at the crack tips, and (ii) the singular integral equations (15) and (16) also appear in the cases (i) of tangential loading of the crack under plane conditions, (ii) of a crack under antiplane conditions and (iii) of a crack inside an anisotropic elastic medium under appropriate conditions.

3. Conclusions

We have previously used the generalized Plemelj formulas (5) and (6) in their simplest form (with \( n = 1 \)) in an equally simple crack problem and we have reduced it to a singular integral
equation with a finite-part (Hadamard-type) integral, i.e. to a hypersingular integral equation. The question which arises is: “Is the use of finite-part integrals necessary in plane and antiplane elasticity problems?” The reply is: “Surely not, but their use is helpful in several cases for a more elegant presentation of the resulting singular integral equations and a more convenient and painless derivation and solution of them.” Exactly the same comments hold true for principal value integrals, so popular in elasticity theory. Neither principal value integrals nor finite-part integrals do result from the physics of the problem under consideration, but in several cases, they prove a useful mathematical tool for its solution. In this context, the author believes that the present results will find wide application in the formulation and solution of plane and antiplane, isotropic and anisotropic elasticity problems.

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References


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