DIVISION OF APPLIED MATHEMATICS AND MECHANICS
DEPARTMENT OF ENGINEERING SCIENCES
SCHOOL OF ENGINEERING, UNIVERSITY OF PATRAS

TECHNICAL REPORT No. TR-1983-Y142, 12 December 1983[1]

A closed-form formula for the critical buckling load of a bar with one end fixed and the other pinned

Eleni G. Anastasselou

Section of Mechanics, Department of Engineering Science,
National Technical University of Athens, GR-157 73 Zographou, Athens, Greece
e-mail: ioakanas@central.ntua.gr

and Nikolaos I. Ioakimidis

Division of Applied Mathematics and Mechanics, Department of Engineering Sciences,
School of Engineering, University of Patras, GR-265 04 Patras, Greece
e-mail: n.ioakimidis@upatras.gr

Abstract The classical problem of elastic buckling of a bar with one end fixed and the other pinned is reconsidered and a closed-form formula for the critical buckling load is derived. This is achieved through the closed-form solution (in terms of two regular integrals) of the transcendental equation $\tan u = u$, to which this problem is reduced. The method of solution of this equation is too simple and based on a generalized form of the Cauchy theorem in complex analysis; yet the sought root of this equation does not contain complex quantities. Finally, numerical results verifying the validity of the derived formula are presented.

Keywords Bars · Buckling · Critical buckling load · Roots · Zeros · Closed-form formulae · Closed-form solutions · Transcendental equations · Cauchy Theorem · Complex analysis

Final publication details[2] This technical report was published in 1984 (with quite minor changes) in *Strojnícky Časopis (Journal of Mechanical Engineering)*, Bratislava. Web address (URL) of the homepage of *Strojnícky Časopis*: http://sc-jme.com. Second web address (URL) at De Gruyter: https://www.degruyter.com/view/j/scjme. The final publication details of this technical report are

- Anastasselou, E. G. and Ioakimidis, N. I., A closed-form formula for the critical buckling load of a bar with one end fixed and the other pinned. *Strojnícky Časopis (Journal of Mechanical Engineering)*, **35** (5), 1984, pp. 621–628.

Therefore, whenever reference to this technical report is to be made, this should be made directly to its above final publication in *Strojnícky Časopis (Journal of Mechanical Engineering)*.

[1] Both the internal and the external links (all appearing in blue) were added by the authors on 31 January 2018 for the online publication of this technical report.

[2] These details were also added by the authors on 31 January 2018 for the online publication of this technical report.

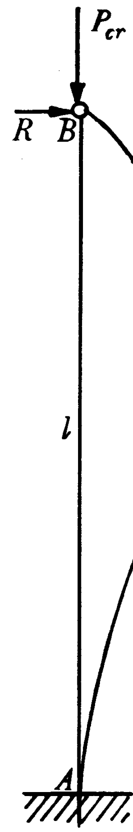

Fig. 1: Geometry of a bar with one end fixed and the other pinned under buckling conditions

1. Introduction

The problem of elastic buckling of a bar with one end fixed and the other pinned (Fig. 1) is a classical problem in the theory of strength of materials, elasticity and elastic stability and it is included even in elementary textbooks on these branches of engineering. The critical buckling load P_{cr} for this problem is of most interest and it is determined by (see, e.g., [1])

$$P_{\mathrm{cr}} = a^2 \frac{EI}{l^2}, \tag{1}$$

where a is the first positive root of the transcendental equation [1]

$$\tan u = u, \tag{2}$$

E is the Young modulus of the elastic material, I the minimum moment of inertia of the cross-section of the bar and l the length of the bar. Although the numerical solution of Eq. (2) is quite possible and the value

$$a \approx 4.493 \;\Rightarrow\; a^2 \approx 20.19 \tag{3}$$

results [1], the closed-form solution of the same equation is also of interest, at least from the theoretical point of view. Then a closed-form formula for the critical buckling load P_{cr} results from Eq. (1).

A sufficiently complicated method for the solution of Eq. (2), based on the theory of the Riemann–Hilbert boundary value problem in complex analysis, was suggested by Burniston and Siewert [2] (although without reference to buckling problems). A much simpler method for the solution of the same equation, based just on a generalized form of the Cauchy theorem in complex analysis [3] and the results of Anastasselou for the solution of transcendental equations [4] is proposed here. The sought root a of Eq. (2) is obtained in Section 2 in terms of two real regular integrals. Numerical results (obtained by applying the Gauss quadrature rule to the evaluation of these integrals) are reported in Section 3 and they verify the validity of this formula.

2. Derivation of the formula

We will determine the first positive root a of Eq. (2). This root is also clearly the root of the equation

$$u = \tan^{-1} u + \pi. \tag{4}$$

Following Burniston and Siewert [2], we put further

$$u = \frac{i}{z}, \quad a = \frac{i}{b} \tag{5}$$

and we seek the root b of the sectionally analytic function

$$F(z) = 1 - z \tanh^{-1} \frac{1}{z} + \pi i z. \tag{6}$$

Next, we consider a smooth contour C surrounding the real interval $[-1,1]$ (the interval of discontinuity of $F(z)$) and the corresponding complex integral

$$I = \int_C (z-b) M(z)\, \mathrm{d}z \tag{7}$$

(in the anticlockwise direction), where

$$M(z) = \frac{1}{F(z)} = \frac{1}{1 - z \tanh^{-1} \dfrac{1}{z} + \pi i z}. \tag{8}$$

By letting $z \to \infty$ and taking into account that

$$M(z) = \frac{1}{\pi i z} + O\left(\frac{1}{z^4}\right), \quad z \to \infty \tag{9}$$

(as is obvious from Eq. (8)) as well as that

$$\int_C z^k \, \mathrm{d}z = 0, \quad k \neq -1, \quad \int_C \frac{1}{z} \, \mathrm{d}z = 2\pi i, \tag{10}$$

we obtain from Eq. (7) (because of the Cauchy theorem)

$$I = -2b. \tag{11}$$

On the other hand, now by letting C shrink to the interval $[-1,1]$ and denoting by $M^{\pm}(x)$ the corresponding values of $M(z)$ as $z \to x \pm 0i$, $x \in (-1,1)$, we obtain

$$I = -\int_{-1}^{1} (x-b)[M^+(x) - M^-(x)]\, \mathrm{d}x. \tag{12}$$

Now we have just to equate the right-hand sides of Eqs. (11) and (12). Then we find

$$b = \frac{\int_{-1}^{1} x[M^+(x) - M^-(x)]\, \mathrm{d}x}{\int_{-1}^{1} [M^+(x) - M^-(x)]\, \mathrm{d}x + 2}. \tag{13}$$

This formula can also be written in a more convenient form. In fact, since

$$\left[\tanh^{-1}\frac{1}{x}\right]^{\pm} = \mp\frac{\pi i}{2} + \tanh^{-1}x, \quad x \in (-1,1), \tag{14}$$

it is clear that

$$M^{+}(x) = \left(1 - x\tanh^{-1}x - \frac{3\pi i x}{2}\right)g^{+}(x),$$
$$M^{-}(x) = \left(1 - x\tanh^{-1}x - \frac{\pi i x}{2}\right)g^{-}(x), \tag{15}$$

where

$$g^{+}(x) = \frac{1}{(1 - x\tanh^{-1}x)^2 + \left(\dfrac{3\pi x}{2}\right)^2},$$
$$g^{-}(x) = \frac{1}{(1 - x\tanh^{-1}x)^2 + \left(\dfrac{\pi x}{2}\right)^2}. \tag{16}$$

By taking into account Eqs. (15) and (16), we directly see that

$$\int_{-1}^{1}[M^{+}(x) - M^{-}(x)]\,\mathrm{d}x = 2\Theta_0,$$
$$\int_{-1}^{1}x[M^{+}(x) - M^{-}(x)]\,\mathrm{d}x = -\pi i\Theta_1, \tag{17}$$

where now

$$\Theta_0 = \int_0^1 (1 - x\tanh^{-1}x)[g^{+}(x) - g^{-}(x)]\,\mathrm{d}x,$$
$$\Theta_1 = \int_0^1 x^2[3g^{+}(x) - g^{-}(x)]\,\mathrm{d}x. \tag{18}$$

Then Eq. (13) takes the form

$$b = -\frac{\pi i\Theta_1}{2(\Theta_0 + 1)}. \tag{19}$$

Finally, because of the second of Eqs. (5), we find for the sought root a of Eqs. (2) and (4)

$$a = -\frac{2(\Theta_0 + 1)}{\pi\Theta_1}, \tag{20}$$

where Θ_0 and Θ_1 are defined by Eqs. (18) and $g^{\pm}(x)$ by Eqs. (16). By inserting the value of a from Eq. (20) into Eq. (1), we obtain the desired closed-form formula for the critical buckling load P_{cr} of the bar of Fig. 1.

3. Numerical results

To verify the validity of our fundamental formula, Eq. (20), we evaluated the integrals Θ_0 and Θ_1 by using the Gauss quadrature rule [5, 6] along the interval $[0,1]$. This formula takes the form [5, 6]

$$\int_0^1 h(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} A_{in}^{*}h(x_{in}^{*}), \tag{21}$$

where the nodes x_{in}^{*} and the corresponding weights A_{in}^{*} are related to the nodes x_{in} and the corresponding weights A_{in} of the Gauss quadrature rule along the interval $[-1,1]$ (tabulated in Reference [6]) through the formulas

$$x_{in}^{*} = \frac{1 + x_{in}}{2}, \quad A_{in}^{*} = \frac{A_{in}}{2}, \quad i = 1,2,\ldots,n. \tag{22}$$

Table 1: Numerical results for the integrals Θ_0 and Θ_1 and the first positive root a of Eq. (2) obtained by using the Gauss quadrature rule with $n = 2, 4, \ldots, 16$ nodes.

n	Θ_0	Θ_1	a
2	−0.261310	−0.118641	3.96376
4	−0.263125	−0.105700	4.43813
6	−0.265686	−0.104399	4.47783
8	−0.265836	−0.104158	4.48724
10	−0.265994	−0.104061	4.49048
12	−0.266070	−0.104018	4.49185
14	−0.266114	−0.103997	4.49251
16	−0.266141	−0.103985	4.49287
"Exact" value of the root a:			4.49341

The obtained numerical results for $n = 2, 4, \ldots, 16$ for the integrals Θ_0 and Θ_1 (defined by Eqs. (18)) and the sought root a of Eq. (2) (determined by Eq. (20)) are displayed in Table 1. From these results we observe their convergence for increasing values of n. Moreover, the Newton–Raphson method was used for the numerical determination of the root a of Eq. (2). The corresponding iterative formula is

$$a_{n+1} = a_n - \frac{\tan a_n - a_n}{\tan^2 a_n} \tag{23}$$

and the resulting "exact" value of a is also displayed in Table 1.

It is also clear that four correct significant digits for the root a were obtained for $n \geq 14$ (after rounding). This is a satisfactory numerical result although the aim of this technical note has been the derivation of a closed-form formula for the critical buckling load P_{cr} in the problem under consideration and not the testing of the Gauss numerical integration rule; the numerical results displayed in Table 1 aim just at the verification of the validity of the aforementioned Eq. (20). On the other hand, the results of this technical note are an illustration to a practical engineering problem of the general method for the closed-form solution of transcendental equations proposed in Reference [4]. The simplicity of this method may lead to its further wide application to other engineering problems in the future.

References[3]

[1] TIMOSHENKO, S. P. and GERE, J. M.: *Theory of Elastic Stability*, 2nd Edn. McGraw-Hill Kogakusha, Tokyo, 1961, p. 53. Web page of the Dover Edn.: http://store.doverpublications.com/0486472078.html

[2] BURNISTON, E. E. and SIEWERT, C. E.: The use of Riemann problem in solving a class of transcendental equations. *Proceedings of the Cambridge Philosophical Society*, **73** (1), 1973, pp. 111–118. https://doi.org/10.1017/S0305004100047526

[3] SVESHNIKOV, A. G. and TIKHONOV, A. N.: *The Theory of Functions of a Complex Variable*. 2nd Edn. Mir Publishers, Moscow, 1978, p. 43. https://archive.org/details/SveshnikovTikhonovTheTheoryOfFunctionsOfAComplexVariable

[3] All the links (external links in blue) in this section were added by the authors on 31 January 2018 for the online publication of this technical report. Moreover, final publication details were added in Reference [4]

[4] ANASTASSELOU, E. G.: *Methods for the Determination of Zeros, Poles and Discontinuity Intervals of Analytic Functions with Applications to Engineering Problems* (in Greek), Doctoral thesis at the National Technical University of Athens. National Technical University of Athens, Athens, 1988.

[5] DAVIS, P. J. and RABINOWITZ, P.: *Numerical Integration*. Blaisdell, Watham, Massachusetts, 1967, p. 33. Web page of the 2nd Edn.: https://www.elsevier.com/books/methods-of-numerical-integration/davis/978-0-12-206360-2. Web page of the Dover Edn. of the 2nd Edn: http://store.doverpublications.com/0486453391.html

[6] STROUD, A. H. and SECREST, DON: *Gaussian Quadrature Formulas*. Prentice Hall, Englewood Cliffs, New Jersey, 1966, p. 99. https://archive.org/details/gaussianquadratu00stro