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A hybrid method for the solution of problems of bending of thin plates

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Abstract An extremely elementary hybrid method for the solution of the problem of thin plates (either finite or infinite with holes) under the action of bending moments and normal forces along the boundary of the plate is proposed. This method is based on the optical method of studying the deformed shape of the boundary of the plate (method of reflected light or 'pseudocaustics') and, subsequently, on the application of elementary analytical techniques from the theory of complex variables. All quantities of interest (deflections, displacements, bending and torsional moments and shear stresses) are determined completely in the whole plate by the present hybrid method.

Keywords Thin plates · Bending · Hybrid methods · Optical methods · Pseudocaustics · Complex variables · Complex potentials · Cauchy formula · Deflections · Displacements · Bending forces · Bending moments · Torsional moments · Torque · Shear stresses

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Therefore, whenever reference to this technical report is to be made, this should be made directly to its above final publication in the *Revue Roumaine des Sciences Techniques – Série de Mécanique Appliquée (Romanian Journal of Technical Sciences – Applied Mechanics)*.

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1. Introduction

We consider the problem of a thin plate, assuming the classical theory valid, loaded along its boundary by bending moments $m(s)$ and bending forces $p(s)$ (perpendicular to the plane of the plate) per unit length of the plate boundary. This is the first fundamental problem in the theory of bending of thin plates. The plate may be finite or infinite, probably with holes. For this interesting problem, reference can be made to the classical books by Timoshenko and Woinowsky-Krieger [1], Green and Zerna [2] and Savin [3]. The important quantities of this problem (in the whole plate) are its deflection (normal displacement) $w(x,y)$, the in-plane displacement components $u(x,y)$ and $v(x,y)$, the bending moments $M_x(x,y)$ and $M_y(x,y)$, the torque $H_{xy}(x,y)$ and the shear stresses per unit length $N_x(x,y)$ and $N_y(x,y)$ [1–3].

In the classical theory of bending of thin plates, it is very well known that we can express all of the above components in terms of two analytic functions, $\phi(z)$ and $\chi(z)$, of the complex variable $z = x + iy$ and their derivatives. This theory is described very briefly in Reference [1] and in much more detail in References [2] and [3]. We will follow the notation of Reference [3], where the exposition of the related theory is elementary. In practice, we normally use the first derivative $\psi(z) = \chi'(z)$ of $\chi(z)$ instead of $\chi(z)$ and this will be done here too. We will not report the related formulae for the aforementioned fundamental quantities of interest, which can be found in Reference [3]. We just mention the arbitrariness of $\phi(z)$ by a quantity of the form $iA_1z + A_2$, where A_1 is a real constant and A_2 is a complex constant, and of $\psi(z)$ by a quantity of the form B, where this is a complex constant.

For the solution of the aforementioned problem, which is in general very difficult (for nonelementary plate geometries), here we propose a very simple mixed method taking into account the shape of the complete boundary L of the plate (after loading) on a screen at a distance Z_0 from the plate, the magnification ratio being equal to λ. The region of the plate near this boundary should have been made reflective with an elementary orthogonal mesh traced on it or, probably more simply, with small reflective areas along the boundary of the plate. This approach, that is to obtain the shape of the deformed boundary of a plane specimen or a plate on a screen, has been used repeatedly in the past (see, e.g., Theocaris and Razem [4] for plane specimens and Theocaris and Gdoutos [5], Ioakimidis and Theocaris [6] for problems of bending of thin plates; the results of Theocaris and Gdoutos [7] are also of interest). The term ‘pseudocaustic’ for the shape of the boundary of the plate on the screen has been adopted although it does not seem successful; this is simply a curve obtained after reflection. In Reference [1] it is noticed that for irregularly shaped plates experimental methods of investigation become more efficient than purely analytical methods. Such methods include electrical strain gauges and extensometers of all kinds, photoelasticity, the interference method, the analogy between plane stress and plate bending and, finally, the use of reflective light through reflective surfaces on the direction of two adjacent light beams for the calculation of the surface curvatures of the plate after bending [1]. The distortion of a luminous rectangular mesh projected on the initially plane surface of the plate may also be used [1].

Yet, the present approach is quite different from the above ones and it is a hybrid method, combining reflection and analytical methods (both completely elementary) and, moreover, it uses information only from the image of the boundary of the plate on the screen and not from all the points of the plate, that is not from the interior points of the plate as well. This seems to be a great advantage of the method. Furthermore, the present method makes use of the efficient complex-variable formulation of the classical theory of bending of thin plates and it is extremely simple, powerful and of general applicability. To the best of this author’s knowledge, the proposed method is completely new.

2. Simple finite plate

The case of a simple finite plate (without holes and points at infinity) is the most elementary one, appropriate for the illustration of the present approach. Without entering into details, which can be found in References [4–7], we have the equations

$$U = \lambda\left(x + C\frac{\partial w}{\partial x}\right), \quad V = \lambda\left(y + C\frac{\partial w}{\partial y}\right), \tag{1}$$

where (x, y) denote points of the boundary L of the plate, (U, V) the corresponding points on the screen, $\partial w/\partial x$ and $\partial w/\partial y$ the first partial derivatives of the deflection $w(x, y)$ of the points of the plate, λ the magnification ratio of the optical set-up and C the related overall constant, which is equal to $-2Z_0/\lambda$, where Z_0 is the distance of the screen from the specimen with the screen lying in front of the specimen. With $z = x + iy$ and $W = U + iV$, Eq. (1) can be written in complex form as

$$W = \lambda\left[z + C\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)\right]. \tag{2}$$

Therefore, by taking into account (directly or from a photograph) the shape of the deformed boundary L of the plate on the screen (such photographs are shown in References [5, 6] and are sufficiently clear; obviously, the photograph in Reference [5] is a 'pseudocaustic' and not a caustic) and knowing the correspondence between the points z and W (by using small reflective surfaces or a mesh as was already mentioned), we determine experimentally the quantity

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = \frac{(W/\lambda) - z}{C} \tag{3}$$

for the points of L. The same quantity is also expressed in terms of the complex potentials $\phi(z)$ and $\psi(z)$ of the theory of thin plates as [3]

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}. \tag{4}$$

Therefore, the latter quantity is also immediately determined from the experiment.

On the other hand, assuming that the loading distribution along L due to the forces $p(s)$ (perpendicular to the plane of the plate) per unit length and the bending moments $m(s)$ also per unit length (where s denotes the arc-length along L) are known, we have for the first fundamental problem of the classical theory of bending of thin plates the following boundary condition [3]:

$$\eta\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = f_1 + if_2 + iA_1z + A_2\,, \tag{5}$$

where A_1 is a real constant and A_2 is a complex constant, due to the arbitrariness of $\phi(z)$ and $\psi(z)$. (For the sake of simplicity, we retained these symbols having been defined for $\phi(z)$ only since their values are of no essential interest and a change of these constants causes only a rigid movement of the plate [3].) Moreover, f_1 and f_2 are easily determined (in terms of $p(s)$ and $m(s)$) loading functions [3] and η is a constant of the material of the plate defined by $\eta = -(3+\nu)/(1-\nu)$ (where ν is the Poisson ratio) [3] and, therefore, it is restricted to the values $-7 < \eta < -3$ since $0 < \nu < 1/2$.

Now, subtracting Eq. (5) from Eq. (4), we obtain

$$\phi(z) = \frac{1}{1-\eta}\left[\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) - (f_1 + if_2) - (iA_1z + A_2)\right]. \tag{6}$$

Since η is a known constant, $(\partial w/\partial x) + i(\partial w/\partial y)$ has been evaluated experimentally from Eq. (3), $f_1 + if_2$ are easily determined and, in this way, known loading distributions along L and A_1 and A_2

are arbitrary constants (the first being real and the second complex) of no interest because of the arbitrariness of $\phi(z)$, Eq. (6) permits us to know the boundary values $\phi(z)$ of this complex potential along L. Then, on the basis of the classical Cauchy formula in complex analysis [8], we have

$$\phi(z_0) = \frac{1}{2\pi i}\int_L \frac{\phi(z)}{z-z_0}\,\mathrm{d}z, \tag{7}$$

where z_0 is an interior point of the plate and $\phi(z)$ was determined from Eq. (6) along L. (The anticlockwise sense along L is assumed in Eq. (7) and in the sequel as usual.)

After the determination of $\phi(z)$, we can also determine $\phi'(z)$ by a differentiation (valid also for Eq. (7)) and, subsequently, use either Eq. (4) or Eq. (5) in order to determine $\psi(z)$ along L. Then we can use Eq. (7) (but with ψ this time) in order to determine this complex potential at any point of the plate.

We easily remark that $\phi(z)$ is arbitrary by a quantity of the form $iA_1z + A_2$ (although, evidently, the values of $A_{1,2}$ differ from those in Eqs. (5) and (6)) and $\psi(z)$ by a constant quantity B as should be the case. The latter fact is clear from Eq. (4), where $\phi(z) + z\bar{\phi}'(\bar{z})$ is not influenced at all by A_1 since this is a real constant and, therefore, the arbitrary term iA_1z does not enter into the values of $\psi(z)$. Moreover, from Eqs. (3) and (6) it is also clear that the origin of the Cartesian coordinate system on the screen (or on the resulting photograph if necessary) causing a change by a complex constant to W and, further, to $\phi(z)$ in Eq. (6) is incorporated into A_2 and, therefore, of no importance at all. We will not give further trivial details, but we will somewhat generalize the previous results.

3. Infinite plate with a hole

This is also an important case in the theory of bending of thin plates having also been considered in detail in Reference [3] and very slightly different from that of the previous section. Denoting by P^* the resultant force on the boundary L of the hole and by M_x^* and M_y^* the corresponding resultant bending moments, for the complex potentials $\phi(z)$ and $\psi(z)$ we have [3]

$$\begin{aligned}\phi(z) &= \frac{1}{2\pi i}\left(\frac{iP^*}{4D}z + \frac{M_x^* + iM_y^*}{4D}\right)\ln z + (E_1 + iA_1)z + A_2 + \tilde{\phi}(z),\\ \psi(z) &= -\frac{1}{2\pi i}\frac{M_x^* - iM_y^*}{4D}\ln z + E_2z + B + \tilde{\psi}(z),\end{aligned} \tag{8}$$

where D is the flexural rigidity of the plate and $E_{1,2}$ are complex constants directly determined from the loading at infinity. The constants $A_{1,2}$ and B are defined exactly as previously and can be ignored in Eqs. (8). Hence, we have to determine $\tilde{\phi}(z)$ and $\tilde{\psi}(z)$, which tend to zero as z tends to infinity and are the only unknown functions in the right-hand sides of Eqs. (8).

To this end, we use the same approach as in the previous section (Eqs. (4) to (6) are still valid), but we subtract from Eq. (6) the first terms in the right-hand side of Eq. (8) in order to stay with $\tilde{\phi}(z)$. Next, this function is determined from Eq. (7) (but now with a minus sign in the integral [8]) and, afterwards, $\phi(z)$ is obtained from the first of Eqs. (8). Analogously, we obtain $\tilde{\psi}(z)$ (using Eq. (4), but with the first terms in the second of Eqs. (8) taken into account, and Eq. (7)) and, further, $\psi(z)$ from the second of Eqs. (8).

The whole approach can be generalized without difficulties to the case of multiple holes of arbitrary shapes as well as to finite plates with one hole or more holes. No essential modification is necessary. On the other hand, quite frequently, we have unloaded holes. (P^*, M_x^*, M_y^*, $p(s)$, $m(s)$ and $f_{1,2}(s)$ vanish in this case.)

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[3] All the links (external links in blue) in this section were added by the author on 28 February 2018 for the online publication of this technical report.