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A new method for the computation of the zeros of analytic functions

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Abstract A new method for the computation of the zeros of analytic functions (or the poles of meromorphic functions) inside or outside a closed contour C in the complex plane is proposed. This method is based on the Cauchy integral formula (in generalized forms) and leads to closed-form formulae for the zeros (or the poles) if they are no more than four. In general, for m zeros (or poles) these can be evaluated as the zeros of a polynomial of degree m. In all cases, complex contour integrals have to be evaluated numerically by using appropriate numerical integration rules. Several practical algorithms for the implementation of the method are proposed and the method of Abd-Elall, Delves and Reid is rederived by two different approaches as one of these algorithms. A numerical application to a transcendental equation appearing in the theory of neutron moderation is also made and numerical results of high accuracy are easily obtained.

Keywords Zeros · Roots · Poles · Residues · Closed-form formulae · Analytical formulae · Analytic functions · Meromorphic functions · Nonlinear equations · Transcendental equations · Cauchy's integral theorem · Cauchy's integral formula · Complex variables · Complex analysis · Complex contour integrals · Numerical integration · Quadrature rules · Neutron moderation

1. Introduction

Several problems in physics and engineering reduce to the determination of the zeros of transcendental functions or, more generally, of analytic functions in the complex plane. This topic also constitutes an active branch of numerical analysis and several methods and numerical algorithms

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have been proposed; see, e.g., Refs. [8, 12]. Moreover, a rather new approach to the determination of the zeros of analytic (or sectionally analytic) functions consists in deriving closed-form formulae for these zeros including contour or real integrals and, afterwards, in using appropriate numerical integration rules for the numerical evaluation of these integrals. This approach seems to be an interesting and efficient one. The first time that this approach was used for the solution of a practical problem is (to these authors' best knowledge) in Ref. [10] by McCune.

Delves and Lyness [7] proposed a generalization of this method based on the following well-known formula in the theory of complex variables:

$$H_n := \frac{1}{2\pi i}\oint_C z^n \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{i=1}^{m} a_i^n, \quad n = 0, 1, \dots, \tag{1}$$

where C is a simple smooth closed contour in the complex plane, $f(z)$ is an analytic function inside C and on C and a_i are the zeros of $f(z)$ inside C. For $n = 0$ Eq. (1) gives the number m of the zeros a_i and it becomes equivalent to the argument principle [2, p. 151]. A modification of this method was recently proposed by Carpentier and Dos Santos [5].

Abd-Elall, Delves and Reid [1] studied further the same method, reported several details of it and discussed it. Moreover, they proposed the use of the formulae

$$K_n := \frac{1}{2\pi i}\oint_C \frac{z^n}{f(z)}\,\mathrm{d}z = \sum_{i=1}^{m} A_i a_i^n, \quad n = 0, 1, \dots, \tag{2}$$

instead of Eqs. (1), where the zeros a_i of $f(z)$ are now assumed simple and A_i are the corresponding residues of the meromorphic function

$$M(z) = \frac{1}{f(z)}, \tag{3}$$

which has a_i as poles of the first order. We call the method based on Eqs. (1) method of Delves and Lyness [7] and the method based on Eqs. (2) method of Abd-Elall, Delves and Reid [1].

Here in Section 2, we will rederive the method of Abd-Elall, Delves and Reid by an approach somewhat different from that used in Ref. [1]. Next, in Section 3, we will propose a new method for the determination of the zeros of analytic functions based on generalized forms of Cauchy's integral formula. Several algorithms result from this new method including that of Abd-Elall, Delves and Reid [1]. In Section 4, we will apply the proposed method to a nontrivial case, the transcendental equation $ze^z = be^b$ appearing in the theory of neutron moderation in nuclear reactors [13]. We will also present numerical results by some of the algorithms of Section 3.

Before proceeding to our results, we feel that we have to make reference to another ingenious related method due to Burniston and Siewert [3, 4]. This method has been used in a very long series of papers (in about twenty papers; see, e.g., Refs. [13–15]) concerning concrete problems of applied mathematics, physics and engineering.

Concluding this introductory section, we can mention that the aim of this technical report is mainly to propose a very general method for the determination of zeros of analytic functions inside or outside a closed contour C in the complex plane by using closed-form formulae based on complex integrals on the contour C. We do not wish to compete the classical numerical methods (such as the Newton–Raphson method in the complex plane) although the present method is frequently superior to these methods even from the numerical point of view. This may happen since these methods may not converge at all or they may present undesirable complications. The closed-form formulae have the following three major advantages compared to numerical methods:

(i) They provide analytical formulae for the sought zeros and, therefore, they are more elegant in their results from the mathematical point of view; e.g., Eq. (20) below is an analytical formula somewhat reminding us of the Newton–Raphson method but it is an exact formula.

(ii) The resulting formulae are valid for all possible values of the parameters involved in the analytic function; on the contrary, this is impossible for numerical methods.

(iii) Similarly, analytical algebraic manipulations can be performed on the resulting formulae for the sought zeros such as series expansions, differentiations, probably under the integral sign, with respect to a parameter, showing the influence of this parameter on the zeros of the analytic function under consideration, etc.; this is also impossible with numerical methods.

Finally, for practical purposes the closed-form formulae derived here can be used for the derivation of numerical results (as will be the case in the application of Section 4) with the additional advantage of the assured convergence of the numerical results contrary to most numerical methods including the Newton–Raphson method.

2. A new derivation of the method of Abd-Elall, Delves and Reid

Here we consider an analytic function $f(z)$ with m simple zeros a_i inside a simple smooth closed contour C in the complex plane and with no zeros on C. We also consider the meromorphic function $M(z)$ defined by Eq. (3). We will determine the simple zeros a_i of $f(z)$ or, equivalently, the simple poles a_i of $M(z)$.

To this end, we construct the polynomial

$$p_m(z) = \sum_{l=0}^{m} b_l z^l = \prod_{i=1}^{m} (z - a_i), \quad b_m = 1. \tag{4}$$

Then, by taking into account the Cauchy integral theorem in complex analysis [2, p. 143], we directly conclude that

$$\oint_C t^k p_m(t) M(t)\, \mathrm{d}t = 0, \quad k = 0, 1, \ldots, m-1, \quad t = x + iy, \tag{5}$$

or, equivalently,

$$\sum_{l=0}^{m} d_{l+k} b_l = 0, \quad k = 0, 1, \ldots, m-1, \tag{6}$$

where

$$d_j := \oint_C t^j M(t)\, \mathrm{d}t, \quad j = 0, 1, \ldots, 2m-1. \tag{7}$$

The coefficients b_l determining the polynomial $p_m(z)$ in Eq. (4), the roots a_i of which are sought, are determined by solving the system of linear equations (6) after the numerical computation of the integrals d_j in Eqs. (7).

The present new derivation of the method of Abd-Elall, Delves and Reid [1] is slightly simpler than the original one since the appearance of the unknown residues A_i in Eqs. (2) is now avoided.

Furthermore, we can mention here that in the special case when $m = 1$, from Eqs. (5) with $p_1(z) = z - a_1$ we directly obtain

$$a_1 = \frac{d_1}{d_0}. \tag{8}$$

Finally, some simple modifications of the method of Abd-Elall, Delves and Reid [1] are evident. At first, in Eqs. (5), we can use a set of m analytic functions $h_k(z)$ inside C and on C instead of the functions z^k now used in Eqs. (5). Of course, in this case, Eqs. (6) should be appropriately modified. An alternative possibility to obtain a system of linear equations for the determination of the coefficients b_l of the polynomial $p_m(z)$ in Eq. (4) is to use a number of different closed contours C_k ($k = 0, 1, \ldots, m-1$) and the formulae

$$\oint_{C_k} p_m(t) M(t)\, \mathrm{d}t = 0, \quad k = 0, 1, \ldots, m-1, \tag{9}$$

instead of Eqs. (5). Then Eqs. (6) will take the forms

$$\sum_{l=0}^{m}\left[b_l \oint_{C_k} t^l M(t)\,\mathrm{d}t\right]=0,\quad k=0,1,\ldots,m-1. \tag{10}$$

But, obviously, in this case, we should be sure that $f(z)$ is analytic in all the regions S_k^+ surrounded by the closed contours C_k and that all the zeros of $f(z)$ lie inside the common region of S_k^+. It is also clear that the existence of multiple zeros of $f(z)$ inside C does not essentially affect the method of Abd-Elall, Delves and Reid [1] provided that $p_m(z)$ is defined by

$$p_m(z):=\prod_{i=1}^{m}(z-a_i)^{j_i} \tag{11}$$

with j_i denoting the multiplicity of each zero a_i. For $m=1$ Eq. (8) does not hold true for $j_1>1$ but an analogous closed-form formula can easily be constructed. Yet, in the case of multiple zeros, their determination by using appropriate derivatives of $f(z)$ with simple zeros only seems to be a good possibility.

3. The proposed method

Now we proceed to the establishment of a new method for the computation of the zeros of $f(z)$. The method of Abd-Elall, Delves and Reid [1] results to be a very special case of this method. At first, let us consider an analytic function $f(z)$ in the interior S^+ of a simple smooth closed contour C and on C with m simple zeros a_i inside C but with no zeros on C. It is also convenient to consider the function $M(z)$ defined by Eq. (3), which is meromorphic in $S^+\cup C$ with m poles a_i of the first order. Then the following formulae are easily seen to hold true [11, p. 276]:

$$\frac{1}{2\pi i}\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t = M(z)-\sum_{i=1}^{m}\frac{A_i}{z-a_i},\quad z\in S^+, \tag{12}$$

$$\frac{1}{2\pi i}\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t = -\sum_{i=1}^{m}\frac{A_i}{z-a_i},\quad z\in S^-, \tag{13}$$

where S^- denotes the part of the complex plane outside C and A_i denote the residues of $M(z)$ corresponding to the poles a_i. The positive direction on C is assumed to be the anticlockwise one. Now defining the new function

$$N(z):=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t - M(z), & z\in S^+,\\[2ex] \dfrac{1}{2\pi i}\displaystyle\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t, & z\in S^-,\end{cases} \tag{14}$$

we can rewrite Eqs. (12) and (13) as

$$\sum_{i=1}^{m}\frac{A_i}{z-a_i}=-N(z),\quad z\in S^+\cup S^-. \tag{15}$$

By differentiating Eq. (15) with respect to z, we also obtain

$$\sum_{i=1}^{m}\frac{A_i}{(z-a_i)^2}=N'(z),\quad z\in S^+\cup S^-. \tag{16}$$

Equation (15) (as well as Eq. (16)) is an identity in the complex plane with the exception of the points of the closed contour C. By using it for several points z_j, we obtain several equations for the determination of the poles a_i of $M(z)=1/f(z)$ or, equivalently, of the zeros a_i of $f(z)$.

The simplest case is that when we have only one zero a_1 with $m = 1$. (Whether this is the case or not can easily be seen by using the argument principle.) Then, by using Eq. (15) for two distinct points z_1 and z_2 in $S^+ \cup S^-$, we obtain

$$\frac{A_1}{z_j - a_1} = -N(z_j), \quad j = 1, 2, \quad z_1 \neq z_2 \,. \tag{17}$$

Of course, if $f(z_j) = 0$, then the sought zero a_1 has been already found. In any case, it is assumed that $f(z_j) \neq 0$ so than $N(z_j)$ is a finite complex number. Now, by solving the system of Eqs. (17) with respect to a_1, we easily find that

$$a_1 = \frac{z_2 N(z_2) - z_1 N(z_1)}{N(z_2) - N(z_1)} \,, \quad z_1 \neq z_2 \,. \tag{18}$$

In this identity, z_1 and z_2 may lie in S^+ or in S^- or one of these two points in S^+ and the other in S^-. This is our first algorithm for the computation of the pole a_1 of $M(z) = 1/f(z)$.

A second algorithm results if we use both Eqs. (15) and (16) again with $m = 1$. Then we directly obtain

$$\frac{A_1}{z - a_1} = -N(z), \quad \frac{A_1}{(z - a_1)^2} = N'(z), \quad z \in S^+ \cup S^-, \tag{19}$$

and further

$$a_1 = z + \frac{N(z)}{N'(z)} \,. \tag{20}$$

This is also an identity in the complex plane valid for every point $z \in S^+ \cup S^-$ different from a_1.

A third algorithm results if we use Eq. (15) for $z \to \infty$. In this special case, we take into account that for $z \to \infty$

$$\frac{1}{z - a} = \sum_{k=0}^{\infty} \frac{a^k}{z^{k+1}} \,. \tag{21}$$

Then Eq. (15) (with the second case of Eq. (14) taken also into account) can be written as

$$\sum_{i=1}^{m} A_i a_i^k = f_k \,, \quad k = 0, 1, \ldots, 2m - 1, \tag{22}$$

where the integrals f_k are defined by

$$f_k := \frac{1}{2\pi i} \oint_C t^k M(t) \, \mathrm{d}t = \frac{d_k}{2\pi i} \,, \quad k = 0, 1, \ldots, 2m - 1, \tag{23}$$

where the quantities d_k were defined by Eqs. (7). Equations (22) with unknowns both the poles a_i of $M(z)$ inside C and the corresponding residues A_i are exactly those appearing during the construction of Gaussian numerical integration rules by the algebraic approach [6, p. 85], where a_i denote the nodes, A_i the corresponding weights and f_k the moments.

For distinct poles a_i (as has been assumed to be the case here) these poles are determined as the roots of a polynomial $p_m(z)$ of the form (4) (with $b_m = 1$) with its coefficients determined from the system of linear equations (6). In this way, we have rederived the method of Abd-Elall, Delves and Reid [1] as a very special case of the method of this section resulting by using Eq. (15) for $z \to \infty$ only. If $m = 1$, Eqs. (22) reduce to

$$A_1 = f_0 \,, \quad A_1 a_1 = f_1 \tag{24}$$

and taking into account Eqs. (23) for $m = 1$, we obtain again Eq. (8). More generally, for $m = 1$ we have

$$a_1 = \frac{f_{j+1}}{f_j} \,, \quad j = 0, 1, \ldots \,. \tag{25}$$

For $m = 1$ we can also obtain a fourth algorithm by using the first of Eqs. (24) for the determination of the residue A_1 together with Eq. (15). Then we find

$$a_1 = z + \frac{f_0}{N(z)}, \quad z \in S^+ \cup S^-, \tag{26}$$

which is also a simple and interesting formula (or, rather, an identity in the complex-plane) for the determination of the pole a_1.

Several more analogous algorithms can be constructed by using Eq. (15) or Eq. (16) too or even further identities resulting by differentiating Eq. (16). For $m = 1$ all the previous algorithms are equally simple and convenient. For $m > 1$ it is advisable to use the third algorithm, i.e. the method of Abd-Elall, Delves and Reid [1], which seems to be the simplest possible method. Of course, in general, we can divide the region S^+ into two or more than two subregions in order to have just one pole of $M(z) = 1/f(z)$ in each one of these subregions.

Now let us proceed to the case when we wish to determine the zeros a_i of $f(z)$ (or, equivalently, the poles a_i of $M(z)$) in the infinite region S^- outside the closed contour C. This case seems not considered so far. We assume that $f(z)$ is analytic in $S^- \cup C$ with a possible pole at infinity. Then $M(z)$ is meromorphic in $S^- \cup C$ with a possible pole at infinity too. If $M(z)$ has a pole of order l at infinity, we denote by $G_\infty(z)$ the principal part of $M(z)$ at infinity, which has the form [11, p. 276]

$$G_\infty(z) = \sum_{j=0}^{l} B_j z^j \tag{27}$$

with the constant term of $M(z)$ at infinity also taken into account in $G_\infty(z)$. Then the following formulae, analogous to Eqs. (12) and (13) hold true [11, p. 276]:

$$\frac{1}{2\pi i}\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t = -M(z) + G_\infty(z) + \sum_{i=1}^{m} \frac{A_i}{z-a_i}, \quad z \in S^-, \tag{28}$$

$$\frac{1}{2\pi i}\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t = G_\infty(z) + \sum_{i=1}^{m} \frac{A_i}{z-a_i}, \quad z \in S^+, \tag{29}$$

the direction on C remaining anticlockwise. We can also define the function

$$N(z) := \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t - M(z) + G_\infty(z), & z \in S^-, \\[2ex] -\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{M(t)}{t-z}\,\mathrm{d}t + G_\infty(z), & z \in S^+, \end{cases} \tag{30}$$

analogously to Eq. (14). Then Eqs. (28) and (29) reduce to Eq. (15); Eq. (16) holds also true. We can further use the algorithms proposed previously without modifications beyond the new definition (30) of $N(z)$ and its consequences on the definition (23) of f_k, which should be modified by taking into account the first case in Eq. (30).

We conclude this section with the remark that if $f(z)$ is a meromorphic function in $S^+ \cup C$ (or in $S^- \cup C$) with zeros a_i and poles g_i in S^+ (or in S^-), then we can apply the previous method twice; once for the location of the zeros a_i by using $M(z) = 1/f(z)$ and once for the location of the poles g_i by using $f(z)$ itself instead of $M(z)$. Finally, as far as the quadrature rules for the computation of contour integrals are concerned, we refer again to the monograph by Davis and Rabinowitz [6, p. 134] as well as to the paper by Lyness and Delves [9].

Table 1
Numerical results for the zero a of Eq. (31) together with Eq. (32) obtained from Eq. (8) (or from Eq. (25) for $j = 0$) by using the trapezoidal quadrature rule on the unit circle for the computation of the integrals d_0 and d_1 with $n = 10, 20, \ldots, 50$ for $b = -1.1, -1.5, -3, -5$ and -10

n	$b = -1.1$	$b = -1.5$	$b = -3$	$b = -5$	$b = -10$
10	−0.75711396	−0.59794328	−0.17823092	−0.034868242	−0.000451324
20	−0.86718231	−0.62531073	−0.17856062	−0.034885768	−0.000454206
30	−0.89280667	−0.62577435	−0.17856063	−0.034885768	−0.000454206
40	−0.90126868	−0.62578239	−0.17856063	−0.034885768	−0.000454206
50	−0.90435836	−0.62578253	−0.17856063	−0.034885768	−0.000454206
Exact values	−0.90625244	−0.62578253	−0.17856063	−0.034885768	−0.000454206

4. A numerical application

Now we consider the classical transcendental equation [15]

$$f(z) := ze^z - c = 0, \tag{31}$$

where c is a known constant. For

$$c = be^b \tag{32}$$

this equation appears in the theory of neutron moderation in nuclear reactors [13]. In this case,

$$M(z) = \frac{1}{ze^z - c} \tag{33}$$

because of Eq. (3). For $b \in (-\infty, -1)$ we seek the zero a of $f(z)$ (or rather the pole a of $M(z)$) in the open interval $(-1, 0)$ [15]. The zero $z = b$ of Eq. (31) is trivial. An investigation of Eq. (31) and its solution by the method of Burniston and Siewert are given in Ref. [15]. Numerical results taking into account Eq. (32) are presented in Ref. [13] (in a corrigendum, p. 247). These numerical results can be verified to be in agreement with those to be displayed below.

Now we proceed to the application of the methods of Section 3. The closed contour C is selected to be the unit circle, $t = e^{i\theta}$ with $\theta \in [0, 2\pi)$. At first, we used Eq. (8) (or Eq. (25) with $j = 0$), that is the method of Abd-Elall, Delves and Reid [1], for the solution of Eq. (31) with the constant c defined by Eq. (32) for $b = -1.1, -1.5, -3, -5$ and -10. The integrals d_0 and d_1, Eqs. (7), or f_0 and f_1, Eqs. (23), were approximated numerically by using the trapezoidal quadrature rule, which is very suitable for the present integrals, on the unit circle C with the polar angle $\theta \in [0, 2\pi)$ as the variable. The polar angles θ_{jn} of the nodes were defined by

$$\theta_{jn} = \frac{(2j-1)\pi}{n}, \quad j = 1, 2, \ldots, n, \quad n = 10, 20, \ldots, 50, \tag{34}$$

so that the corresponding nodes $t_{jn} = e^{i\theta_{jn}}$ in the complex plane $z = x + iy$ lie symmetrically with respect to the real axis Ox. Then the trapezoidal quadrature rule

$$\frac{1}{2\pi i} \oint_C g(t)\,\mathrm{d}t \approx \frac{1}{n} \sum_{j=1}^{n} t_{jn}\, g(t_{jn}), \quad t = e^{i\theta}, \tag{35}$$

is considerably simplified in our case, where n is an even positive integer, since $M(\bar{z}) = \overline{M(z)}$ for real values of c and, therefore, essentially only the $n/2$ nodes in the upper half-plane are used. This remark concerns all the computations of integrals in the present section.

Table 2
Numerical results analogous to those of Table 1 but obtained from Eq. (26) for $z = -5, -2, -0.6, 0, 0.6$ and 2

n	$b = -1.1$	$b = -1.5$	$b = -3$	$b = -5$	$b = -10$
			$z = -5$		
10	−0.74348094	−0.59071210	−0.17777973	−0.034761402	−0.000380117
20	−0.86482634	−0.62519281	−0.17856061	−0.034885767	−0.000454206
30	−0.89208988	−0.62577231	−0.17856063	−0.034885768	−0.000454206
40	−0.90101440	−0.62578236	−0.17856063	−0.034885768	−0.000454205
50	−0.90426330	−0.62578253	−0.17856063	−0.034885769	−0.000454206
			$z = -2$		
10	−0.69062384	−0.55145163	−0.16065150	−0.019967588	+0.014072054
20	−0.85661741	−0.62449025	−0.17854272	−0.034871284	−0.000440108
30	−0.88966431	−0.62576006	−0.17856061	−0.034885754	−0.000454192
40	−0.90016208	−0.62578214	−0.17856063	−0.034885769	−0.000454206
50	−0.90394575	−0.62578253	−0.17856063	−0.034885769	−0.000454206
			$z = -0.6$		
10	−0.83505917	−0.62486491	−0.18265544	−0.040875805	−0.006947316
20	−0.88405206	−0.62576813	−0.17858543	−0.034922579	−0.000494131
30	−0.89823249	−0.62578229	−0.17856078	−0.034885991	−0.000454447
40	−0.90322999	−0.62578253	−0.17856063	−0.034885769	−0.000454207
50	−0.90509668	−0.62578253	−0.17856063	−0.034885768	−0.000454205
			$z = 0$		
10	−0.79805218	−0.61437995	−0.17854100	−0.034885646	−0.000454205
20	−0.87516796	−0.62558577	−0.17856063	−0.034885768	−0.000454206
30	−0.89530870	−0.62577912	−0.17856063	−0.034885768	−0.000454206
40	−0.90216501	−0.62578247	−0.17856063	−0.034885768	−0.000454206
50	−0.90469465	−0.62578253	−0.17856063	−0.034885768	−0.000454206
			$z = 0.6$		
10	−0.78404193	−0.60846272	−0.17648017	−0.032886521	+0.001518940
20	−0.87241000	−0.62549950	−0.17854837	−0.034873581	−0.000442165
30	−0.89443255	−0.62577771	−0.17856055	−0.034885695	−0.000454133
40	−0.90184958	−0.62578245	−0.17856063	−0.034885768	−0.000454205
50	−0.90457609	−0.62578253	−0.17856063	−0.034885768	−0.000454206
			$z = 2$		
10	−0.77289836	−0.60515299	−0.17863012	−0.035134913	−0.000717681
20	−0.87008008	−0.62542871	−0.17856083	−0.034886017	−0.000454463
30	−0.89370146	−0.62577640	−0.17856063	−0.034885769	−0.000454206
40	−0.90158766	−0.62578243	−0.17856063	−0.034885768	−0.000454206
50	−0.90447782	−0.62578253	−0.17856063	−0.034885768	−0.000454206

Table 3
Numerical results analogous to those of Table 2 but obtained from Eq. (20) again for $z = -5, -2, -0.6, 0, 0.6$ and 2

n	$b = -1.1$	$b = -1.5$	$b = -3$	$b = -5$	$b = -10$
			$z = -5$		
10	−0.72795656	−0.58151832	−0.17678142	−0.034374223	−0.000071123
20	−0.86230149	−0.62504539	−0.17856060	−0.034885768	−0.000454205
30	−0.89133212	−0.62576976	−0.17856063	−0.034885768	−0.000454206
40	−0.90074679	−0.62578231	−0.17856063	−0.034885768	−0.000454206
50	−0.90416341	−0.62578253	−0.17856063	−0.034885769	−0.000454206
			$z = -2$		
10	−0.57273411	−0.43667830	−0.05760247	+0.087119513	+0.122712790
20	−0.84241884	−0.62229467	−0.17828848	−0.034631844	−0.000200493
30	−0.88571478	−0.62572087	−0.17856020	−0.034885382	−0.000453820
40	−0.89880055	−0.62578146	−0.17856063	−0.034885768	−0.000454206
50	−0.90344198	−0.62578252	−0.17856063	−0.034885769	−0.000454206
			$z = -0.6$		
10	−0.86377622	−0.62569050	−0.14607564	+0.031657077	+0.076734189
20	−0.89318996	−0.62578133	−0.17819995	−0.034170107	+0.000368753
30	−0.90141738	−0.62578252	−0.17855740	−0.034879350	−0.000446823
40	−0.90441230	−0.62578253	−0.17856060	−0.034885717	−0.000454147
50	−0.90554617	−0.62578253	−0.17856063	−0.034885768	−0.000454205
			$z = 0$		
10	−0.82509309	−0.62106140	−0.17855946	−0.034885767	−0.000454206
20	−0.88134680	−0.62570046	−0.17856063	−0.034885768	−0.000454206
30	−0.89732509	−0.62578111	−0.17856063	−0.034885768	−0.000454206
40	−0.90289741	−0.62578251	−0.17856063	−0.034885768	−0.000454206
50	−0.90497085	−0.62578253	−0.17856063	−0.034885768	−0.000454206
			$z = 0.6$		
10	−0.82182305	−0.64105511	−0.20357974	−0.055241790	−0.019479676
20	−0.87698827	−0.62592498	−0.17886862	−0.035135948	−0.000688056
30	−0.89585303	−0.62578253	−0.17856345	−0.034888060	−0.000456348
40	−0.90236146	−0.62578251	−0.17856065	−0.034885787	−0.000454223
50	−0.90476861	−0.62578253	−0.17856063	−0.034885768	−0.000454206
			$z = 2$		
10	−0.78712901	−0.61178140	−0.18129765	−0.037993175	−0.003625028
20	−0.87273279	−0.62551959	−0.17856563	−0.034891333	−0.000459883
30	−0.89453334	−0.62577793	−0.17856063	−0.034885776	−0.000454214
40	−0.90188578	−0.62578245	−0.17856063	−0.034885768	−0.000454206
50	−0.90458969	−0.62578253	−0.17856063	−0.034885768	−0.000454206

The numerical results obtained by the method of Abd-Elall, Delves and Reid [1] are displayed in Table 1 together with the "exact" values of the zeros a obtained by an improvement of the corresponding approximate values (for $n = 50$) by the Newton–Raphson method. The results of Table 1 are quite satisfactory especially for $b = -3, -5$ and -10. The worst results were obtained for $b = -1.1$. This can be explained since b is also a pole of the integrand $M(z)$ outside but near the contour C and the contribution of this pole b has been ignored in the numerical integrations.

In Table 2, we present analogous approximate results (obtained by using again the trapezoidal quadrature rule) when Eq. (26) is used for the computation of the zero a, where f_0 was computed on the basis of the first of Eqs. (23) and $N(z)$ from Eq. (14). Six values of z were used in Eq. (26): $z = -5, -2, -0.6, 0, 0.6$ and 2: three values inside the unit circle C (whence the first case of Eq. (14) holds true) and three values outside this circle (whence the second case of Eq. (14) is valid). The numerical results of Table 2 are also quite satisfactory and completely comparable with the corresponding results of Table 1 and the "exact" values of a displayed in that table.

Finally, analogous numerical results, but based on the use of Eq. (20), are displayed in Table 3 for the same values of z. These results show that this formula, Eq. (20), can also be successfully used for the computation of the zero a of $f(z)$. Finally, we can add that we avoided to use complex values of z both in Eq. (26) and in Eq. (20) in order to avail ourselves of the fact that $M(\bar{z}) = \overline{M(z)}$ (as was already mentioned) and from the practical point of view in order to reduce the number of nodes to $n/2$ during the computations of the corresponding contour integrals.

5. Generalizations

The present method for the computation of the zeros of analytic functions can be generalized to more complicated cases. We report in brief four such possible generalizations:

(i) To determine the zeros of $f(z)$ inside a multiply-connected region S, e.g. one between two nonintersecting closed contours C_1 and C_2; this case is a simple one.

(ii) To determine the zeros of periodic, doubly-periodic and, more generally, automorphic functions $f(z)$ inside a fundamental region S^+ bounded by a fundamental closed contour C; this generalization also seems to be a rather simple one.

(iii) To determine the zeros of Cauchy-type principal value integrals or of their derivatives on the arc L where they are defined; this is a more difficult problem and the results of the previous sections should be further generalized.

(iv) To determine the solutions of a system of two equations of the form $f_1(z_1, z_2) = 0$ and $f_2(z_1, z_2) = 0$, where $z_1 \in S_1$ and $z_2 \in S_2$; this problem seems to be sufficiently difficult but simultaneously of sufficient interest.

References[2]

[1] Abd-Elall, L. F., Delves, L. M. and Reid, J. K., A numerical method for locating the zeros and poles of a meromorphic function. In *Numerical Methods for Nonlinear Algebraic Equations* (Rabinowitz, P., editor). Gordon and Breach, London, 1970, chap. 3, pp. 47–59.

[2] Ahlfors, L. V., *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, second edition. McGraw-Hill, New York, 1966. Web page of the first edition, 1953: https://archive.org/details/complexanalysisi00ahlf

[3] Burniston, E. E. and Siewert, C. E., The use of Riemann problems in solving a class of transcendental equations. *Mathematical Proceedings of the Cambridge Philosophical Society*, **73** (1), 111–118 (1973). https://doi.org/10.1017/S0305004100047526

[2] All the links (external links in blue) in this section were added by the authors on 13 April 2018 for the online publication of this technical report.

[4] Burniston, E. E. and Siewert, C. E., Exact analytical solutions of the transcendental equation $\alpha \sin \zeta = \zeta$. *SIAM Journal on Applied Mathematics*, **24** (4), 460–466 (1973). https://doi.org/10.1137/0124048

[5] Carpentier, M. P. and Dos Santos, A. F., Solution of equations involving analytic functions. *Journal of Computational Physics*, **45** (2), 210–220 (1982). https://doi.org/10.1016/0021-9991(82)90117-6

[6] Davis, P. J. and Rabinowitz, P., *Methods of Numerical Integration*, first edition. Academic Press, New York, 1975. Web page of the second edition, 1984: https://www.elsevier.com/books/methods-of-numerical-integration/davis/978-0-12-206360-2. Web page of the Dover edition of the second edition: http://store.dover publications.com/0486453391.html

[7] Delves, L. M. and Lyness, J. N., A numerical method for locating the zeros of an analytic function. *Mathematics of Computation*, **21** (100), 543–560 (1967). https://doi.org/10.1090/S0025-5718-1967-0228165-4

[8] Householder, A. S., *The Numerical Treatment of a Single Nonlinear Equation*. McGraw-Hill, New York, 1970.

[9] Lyness, J. N. and Delves, L. M., On numerical contour integration round a closed contour. *Mathematics of Computation*, **21** (100), 561–577 (1967). https://doi.org/10.2307/2005000

[10] McCune, J. E., Exact inversion of dispersion relations. *The Physics of Fluids*, **9** (10), 2082–2084 (1966). https://doi.org/10.1063/1.1761574

[11] Muskhelishvili, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity: Fundamental Equations, Plane Theory of Elasticity, Torsion and Bending* (second English edition; translation of the 4th Russian edition, Moscow, 1954; 5th Russian edition: Nauka, Moscow, 1966). P. Noordhoff, Groningen, The Netherlands, 1963. http://www.springer.com/gp/book/9789001607012

[12] Rabinowitz, P. (editor), *Numerical Methods for Nonlinear Algebraic Equations*. Gordon and Breach, London, 1970.

[13] Siewert, C. E. and Burkart, A. R., An asymptotic solution in the theory of neutron moderation. *Nuclear Science and Engineering*, **54** (4), 455–456 (1974). https://doi.org/10.13182/NSE74-A23440

[14] Siewert, C. E. and Burniston, E. E., An explicit closed-form result for the discrete eigenvalue in studies of polarized light. *The Astrophysical Journal*, **173**, 405–406 (1972). http://adsabs.harvard.edu/full/1972ApJ...173..405S

[15] Siewert, C. E. and Burniston, E. E., Exact analytical solutions of $ze^z = a$. *Journal of Mathematical Analysis and Applications*, **43** (3), 626–632 (1973). https://doi.org/10.1016/0022-247X(73)90281-3