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A modification of the generalized airfoil equation and the corresponding numerical methods

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Abstract The two-dimensional problem of steady, inviscid, irrotational, subsonic flow around a straight or curvilinear thin airfoil or an array of such airfoils inside a wind tunnel is generally reduced to a one-dimensional Cauchy type real or complex singular integral equation called generalized airfoil equation. Here a new form of this equation is suggested (with no change in the unknown function) with index equal to 1 instead of 0. The new equation is supplemented by an integral condition assuring the uniqueness of its solution. This modification of the generalized airfoil equation permits the application of the theoretical results and the algorithms for the numerical solution of Cauchy type singular integral equations with index equal to 1 (mainly appearing in crack problems in the theory of plane elasticity) to the generalized airfoil equation and it establishes the relationship between crack and airfoil problems. Moreover, it permits the utilization of the classical Chebyshev polynomials instead of the airfoil polynomials. Three applications are also made and numerical results are presented.

Keywords Steady flow · Inviscid flow · Irrotational flow · Subsonic flow · Thin airfoils · Cracks · Wind tunnels · Airfoil equation · Generalized airfoil equation · Airfoil polynomials · Chebyshev polynomials · Cauchy type integrals · Cauchy type singular integral equations · Numerical methods · Numerical integration · Galerkin method · Gauss–Chebyshev method · Lobatto–Chebyshev method

1. Introduction

The airfoil equation is a classical Cauchy type singular integral equation in fluid mechanics and hydrodynamics. It results in the two-dimensional problem of steady flow of an ideal fluid past

[1]Both the internal and the external links (all appearing in blue) were added by the author on 19 April 2018 for the online publication of this technical report.

a thin almost straight airfoil and it has the form

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,\frac{g(t)}{t-x}\,\mathrm{d}t = f(x), \quad -1 < x \leq 1, \tag{1}$$

where $g(t)$ is the unknown function and $f(x)$ is a known function. For a derivation of Eq. (1) (see, e.g., Ref. [1]). In more complicated cases, such as those of curvilinear airfoils (see, e.g., Ref. [2]), arrays of airfoils, airfoils inside tunnels (see, e.g., Refs. [3–5]), Eq. (1) takes the more general form

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{\pi(t-x)}+k(t,x)\right]g(t)\,\mathrm{d}t = f(x), \quad -1 < x \leq 1, \tag{2}$$

where $k(t,x)$ is an appropriate known kernel. For curvilinear airfoils we have a complex singular integral equation of the form (2) or, better, a pair of two real singular integral equations of this form. Moreover, for oscillating airfoils the kernel $k(t,x)$ has also a weakly singular part of the form $c\ln|t-x|$, where c is an arbitrary constant [3–5]. For convenience, here we will consider only a simple real singular integral equation of the form (2) with a regular kernel $k(t,x)$. Our results can easily be generalized to systems of singular integral equations, to complex singular integral equations and to singular integral equations with a logarithmic singularity in the regular part $k(t,x)$ of their kernels.

The airfoil equation (1) possesses the closed-form solution (see, e.g., Refs. [1–3])

$$g(t) = -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,\frac{f(x)}{x-t}\,\mathrm{d}x, \quad -1 \leq t < 1. \tag{3}$$

On the contrary, in general, the generalized airfoil equation (2) does not possess a closed-form solution. For its numerical solution several numerical methods have been proposed such as the collocation method [3–8], the Galerkin method [9–11] and the quadrature method [10, 12–15].

All of these methods are based on the use of the Jacobi polynomials of degree n, i.e. on the polynomials $P_n^{(1/2,-1/2)}(x)$ and $P_n^{(-1/2,1/2)}(x)$, or, better, of the so-called airfoil polynomials [3–8]

$$t_n(x) = \cos\frac{\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}}, \tag{4}$$

$$u_n(x) = \sin\frac{\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}, \tag{5}$$

where

$$x = \cos\theta, \quad 0 \leq \theta \leq \pi. \tag{6}$$

The polynomials $t_n(x)$ are called airfoil polynomials of the second kind or downwash polynomials whereas the polynomials $u_n(x)$ are called airfoil polynomials of the first kind or pressure polynomials [3–5]. Moreover, these polynomials satisfy the orthogonality relations

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,t_m(x)t_n(x)\,\mathrm{d}x = \delta_{mn}, \tag{7}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,u_m(x)u_n(x)\,\mathrm{d}x = \delta_{mn}, \tag{8}$$

where the symbol δ_{mn} denotes Kronecker's delta. They also satisfy the recurrence relations

$$t_{n+1}(x) = 2xt_n(x) - t_{n-1}(x), \tag{9}$$

$$u_{n+1}(x) = 2xu_n(x) - u_{n-1}(x) \tag{10}$$

and they have the following Hilbert transforms:

$$\frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \frac{t_n(x)}{x-t} \, \mathrm{d}x = u_n(t), \tag{11}$$

$$\frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{u_n(t)}{t-x} \, \mathrm{d}t = -t_n(x). \tag{12}$$

These properties of the airfoil polynomials are considered in detail in Refs. [3–5].

Here we will transform the singular integral equation (2) (the generalized airfoil equation), which has index κ equal to 0, to a singular integral equation with index κ equal to 1 supplemented by an integral condition. This is the situation appearing in crack problems of plane elasticity (see, e.g., Refs. [10, 15]). In this way, the relationship between airfoil and crack problems will be established. Moreover, most results on the algorithms for the numerical solution of singular integral equations concern such equations with index κ equal to 1 because of their appearance in crack problems (see, e.g., Refs. [16, 17]). Therefore, these results become useful also for the generalized airfoil equation in its modified form. From the practical point of view, we will be able to apply the numerical methods already available for crack problems to airfoil problems.

Finally, we will see that we can use the Chebyshev polynomials of the first and the second kind and degree n (see, e.g., Ref. [18])

$$T_n(x) = \cos n\theta, \tag{13}$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{14}$$

respectively (with the angle θ defined by Eq. (6)) instead of the airfoil polynomials $u_n(x)$ and $t_n(x)$, respectively. The Chebyshev polynomials are more popular than the airfoil polynomials and this may be an interesting result. The relationships between the original generalized airfoil equation and its modified form as well as between the corresponding sets of orthogonal polynomials will be investigated in detail, two simple applications will be made and numerical results will be displayed. Generalizations and further applications of the present results are also quite possible.

At this point we can mention that Chebyshev polynomials have also been used in the airfoil equation by Lan for the reduction of the chordwise vortex integral to a finite sum through a modified trapezoidal rule and the theory of Chebyshev polynomials [19]. Moreover, Lan showed how to extract the leading-edge suction parameter from the solution of the airfoil equation [19].

2. Modification of the generalized airfoil equation

At first, we notice that the generalized airfoil equation (2) can also be written as

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \left[\frac{1}{\pi(t-x)} + k(t,x) \right] h(t) \, \mathrm{d}t = f(x), \quad -1 \leq x \leq 1, \tag{15}$$

where the new unknown function $h(t)$ is related to the initial unknown function $g(t)$ by

$$h(t) = (1-t)g(t) \tag{16}$$

and it should satisfy the condition (Kutta condition)

$$h(1)=0 \tag{17}$$

for the uniqueness of the solution of Eq. (15). On the other hand, a crack problem is reduced to a singular integral equation of the form [10, 15]

$$\fint_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\left[\frac{1}{\pi(t-x)}+k(t,x)\right]g(t)\,\mathrm{d}t=f(x),\quad -1\le x\le 1, \tag{18}$$

accompanied by the integral condition of single-valuedness of displacements on the crack

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,g(t)\,\mathrm{d}t=0. \tag{19}$$

Although the singular integral equations (15) and (18) coincide, the supplementary conditions (17) and (19), respectively, are quite different. Moreover, the change (16) of the unknown function in the airfoil equation causes a loss of information about this function near the point $t=1$ (the trailing edge of the airfoil) being simultaneously inconvenient itself. For these reasons the aforementioned modification of the singular integral equation (2), well known in the literature (see, e.g., Ref. [1]), is not adopted here and an alternative possibility is suggested.

Here we take into account that

$$\frac{1}{\pi}\fint_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,\frac{g(t)}{t-x}\,\mathrm{d}t=\frac{1}{\pi}(1-x)\fint_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,\frac{g(t)}{t-x}\,\mathrm{d}t-\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,g(t)\,\mathrm{d}t, \tag{20}$$

as well as that

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,k(t,x)g(t)\,\mathrm{d}t=(1-x)\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}K(t,x)g(t)\,\mathrm{d}t+\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,k(t,1)g(t)\,\mathrm{d}t, \tag{21}$$

where

$$K(t,x):=-(1-t)\,\frac{k(t,x)-k(t,1)}{x-1}. \tag{22}$$

Evidently,

$$\lim_{x\to 1}K(t,x)=-(1-t)\,\frac{\partial}{\partial x}k(t,x)\Big|_{x=1} \tag{23}$$

and we assume that this partial derivative exists.

Now, because of Eqs. (20) and (21), the singular integral equation (2) can be written as

$$(1-x)\fint_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\left[\frac{1}{\pi(t-x)}+K(t,x)\right]g(t)\,\mathrm{d}t+\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\left[-\frac{1}{\pi}+(1-t)\,k(t,1)\right]g(t)\,\mathrm{d}t=f(x). \tag{24}$$

This equation holds true for $x\to 1$. Then we find

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\left[-\frac{1}{\pi}+(1-t)\,k(t,1)\right]g(t)\,\mathrm{d}t=f(1). \tag{25}$$

This integral condition permits us to write Eq. (24) as

$$\fint_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\left[\frac{1}{\pi(t-x)}+K(t,x)\right]g(t)\,\mathrm{d}t=F(x), \tag{26}$$

where

$$F(x):=-\frac{f(x)-f(1)}{x-1}. \tag{27}$$

Obviously,

$$\lim_{x\to 1} F(x) = -f'(1) \tag{28}$$

and this derivative is assumed to exist.

The singular integral equation (26) coincides with the singular integral equation (18), which is valid for crack problems. Moreover, the integral condition (25) is a generalization of the condition (19), which is valid for crack problems. These two conditions coincide if $k(t,x) \equiv 0$, that is for the original singular integral equation (1). The singular integral equation (26) and the supplementary condition (25) constitute the modification of the generalized airfoil equation (2) proposed here. Finally, we observe that if the kernel $k(t,x)$ presents a logarithmic singularity of the form $c \ln|t-x|$, then the same behaviour is transferred to the modified kernel $K(t,x)$ defined by Eq. (22).

3. Numerical methods

For the numerical solution of the singular integral equation (26) supplemented by the condition (25), we can apply any of the methods available in the literature for singular integral equations with index κ equal to 1 [9–12, 14–17]. For example, if we use the well-known Gauss–Chebyshev method [10, 12], we obtain the following approximate system of linear algebraic equations:

$$\frac{\pi}{n}\sum_{i=1}^{n}\Big[\frac{1}{\pi(t_i-x_k)}+K(t_i,x_k)\Big]g_n(t_i) = F(x_k), \quad k=1,2,\dots,n-1, \tag{29}$$

$$\frac{\pi}{n}\sum_{i=1}^{n}\Big[-\frac{1}{\pi}+(1-t_i)\,k(t_i,1)\Big]g_n(t_i) = f(1) \tag{30}$$

from Eqs. (26) and (25), respectively. In these equations, the nodes t_i are the roots of the Chebyshev polynomial of the first kind and degree n, $T_n(x)$, whereas the collocation points x_k are the roots of the Chebyshev polynomial of the second kind and degree $n-1$, $U_{n-1}(x)$ [10, 12]. Moreover, the symbols $g_n(t_i)$ denote approximations to the exact values $g(t_i)$ of the unknown function $g(t)$ at the nodes t_i.

We can also mention the formulae

$$t_n(x) = T_n(x) - (1-x)U_{n-1}(x), \tag{31}$$

$$u_n(x) = T_n(x) + (1+x)U_{n-1}(x) \tag{32}$$

relating the airfoil polynomials $t_n(x)$ and $u_n(x)$ with the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$. These formulae are directly deduced from Eqs. (4) and (5) on the basis of Eqs (13) and (14). The Chebyshev polynomials also satisfy the following formulae [20]:

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,\frac{U_{n-1}(x)}{x-t}\,\mathrm{d}x = -T_n(t), \tag{33}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,\frac{T_n(t)}{t-x}\,\mathrm{d}t = U_{n-1}(x). \tag{34}$$

These formulae are analogous to Eqs. (11) and (12) for the airfoil polynomials $t_n(x)$ and $u_n(x)$.

At this point, we wish to mention that the numerical results that we obtain (under analogous conditions) from the numerical solution of Eqs. (26) and (25) in general do not coincide with the corresponding results that we obtain directly from the solution of Eq. (2). In order to see this, we can apply the weighted Galerkin method [6, 9–11, 16] to the classical airfoil equation (1) as well as to the equivalent equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,\frac{g(t)}{t-x}\,\mathrm{d}t = F(x) \tag{35}$$

resulting from Eq. (26) for $K(t,x) \equiv 0$ and accompanied by the condition

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, g(t)\,\mathrm{d}t = -f(1) \tag{36}$$

resulting from Eq. (25) for $k(t,x) \equiv 0$.

In order to directly solve the singular integral equation (1) by the weighted Galerkin method, we assume the unknown function $g(t)$ to be approximated by the function

$$g_{n1}(t) = \sum_{i=0}^{n} b_i u_i(t). \tag{37}$$

Then on the basis of Eqs. (7) and (12), we find that

$$b_i = -c_i\,, \tag{38}$$

where c_i are the coefficients in the series expansion

$$f(x) = \sum_{i=0}^{\infty} c_i t_i(x) \tag{39}$$

of the function $f(x)$ in downwash polynomials. These coefficients are determined by

$$c_i = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, f(x)\, t_i(x)\,\mathrm{d}x. \tag{40}$$

Now, if we know the coefficients d_i in the series expansion

$$f(x) = 2\sum_{i=0}^{\infty}{}' d_i T_i(x) \tag{41}$$

of the function $f(x)$ in Chebyshev polynomials $T_i(x)$, whence

$$d_i = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, f(x) T_i(x)\,\mathrm{d}x, \tag{42}$$

then on the basis of Eqs. (31), (40) and (42), it is easy to find that

$$c_i = d_{i+1} + d_i\,. \tag{43}$$

On the other hand, when we solve Eqs. (35) and (36) by using the weighted Galerkin method [6, 9–11, 16], we approximate the unknown function $g(t)$ by the function

$$g_{n2}(t) = \sum_{i=0}^{n} e_i T_i(t). \tag{44}$$

From Eq. (36) we directly find that

$$e_0 = -f(1) \tag{45}$$

while Eq. (35) yields

$$e_i = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}\, F(x)\, U_{i-1}(x)\,\mathrm{d}x, \quad i = 1, 2, \dots\,. \tag{46}$$

Because of Eq. (45), Eq. (44) can be rewritten as

$$g_{n2}(t) = -f(1) + \sum_{i=1}^{n} e_i T_i(t). \tag{47}$$

Furthermore, by taking into account Eqs. (27), (38), (40) and (46) as well as that

$$T_i(x) = U_i(x) - xU_{i-1}(x), \tag{48}$$

it is easy to find that

$$b_0 = -f(1) - \frac{e_1}{2}, \quad b_i = \frac{e_i - e_{i+1}}{2}, \quad i = 1, 2, \dots . \tag{49}$$

Now Eq. (37), because of Eqs. (32) and (49), is written as

$$g_{n1}(t) = -f(1) - \frac{e_1}{2} + \frac{1}{2}\sum_{i=1}^{n}(e_i - e_{i+1})\,[T_i(t) + (1+t)U_{i-1}(t)]. \tag{50}$$

Finally, from Eqs. (47) and (50) we easily conclude that

$$g_{n2}(t) - g_{n1}(t) = \frac{e_{n+1}}{2}\,u_n(t). \tag{51}$$

This is the difference between the approximate solutions of Eqs. (35, 36) and Eq. (1) by the corresponding weighted Galerkin methods with the same value of n, which is equal to the degree of the polynomials $g_{n1}(t)$ and $g_{n2}(t)$.

4. Applications

As a first application, we consider the airfoil equation (1) with

$$-f(x) = 16x^4 = t_4(x) + t_3(x) + 4t_2(x) + 4t_1(x) + 6t_0(x). \tag{52}$$

Then from Eq. (27) we easily obtain for the function $F(x)$

$$F(x) = 16(x^3 + x^2 + x + 1) = 2[U_3(x) + 2U_2(x) + 6U_1(x) + 10U_0(x)]. \tag{53}$$

In Table 1, we present the values of the coefficients b_i and e_i in the approximations (37) and (44), respectively, to the unknown function $g(t)$ in the singular integral equation (1) together with the corresponding explicit formulae for these approximations $g_{n1}(t)$ and $g_{n2}(t)$ and the numerical values of these functions for $t = -1$, 0 and 1. These numerical results were obtained in accordance with the developments of the previous section. From the results of Table 1 we observe that for $n = 4$ the approximations $g_{n1}(t)$ and $g_{n2}(t)$ to $g(t)$ coincide with $g(t)$. Moreover, Eq. (51) is seen to hold true for all values of n as is clear from the expressions of the difference $g_{n2}(t) - g_{n1}(t)$ displayed in the last part of Table 1 together with its values for $t = -1$, $t = 0$ and $t = 1$.

Next, in Table 2 and in Table 3, we present numerical results analogous to those of Table 1 but now for the functions

$$f(x) = (1-x)e^x \tag{54}$$

in the singular integral equation (1). Then, because of Eq. (27),

$$F(x) = e^x. \tag{55}$$

More specifically, in Table 2, the values of the coefficients

$$\gamma_i = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,e^x\,T_i(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos i\theta\,\mathrm{d}\theta = I_i(1), \tag{56}$$

where $I_i(x)$ denote the well-known modified Bessel functions of imaginary argument [18], are presented. The numerical values of γ_i for $i = 0, 1, \dots, 10$ were obtained from Ref. [20]. Moreover, the values of e_i determined from Eq. (45) and the equations

$$e_i = \gamma_{i-1} - \gamma_{i+1}, \quad i = 1, 2, \dots, \tag{57}$$

Table 1
Results for the approximate solutions $g_{n1}(t)$ and $g_{n2}(t)$ of the airfoil equation (1) for $f(x) = -16x^4$ and $n = 0, 1, \ldots, 4$ by the methods of Section 3

		$g_{n1}(t)$			
i, n	b_i	general formula	$t=-1$	$t=0$	$t=1$
0	6	6	6	6	6
1	4	$8t+10$	2	10	18
2	4	$16t^2+16t+6$	6	6	38
3	1	$8t^3+20t^2+12t+5$	5	5	45
4	1	$16t^4+16t^3+8t^2+8t+6$	6	6	54
Theoretical results		$16t^4+16t^3+8t^2+8t+6$	6	6	54
		$g_{n2}(t)$			
i, n	e_i	general formula	$t=-1$	$t=0$	$t=1$
0	16	16	16	16	16
1	20	$20t+16$	-4	16	36
2	12	$24t^2+20t+4$	8	4	48
3	4	$16t^3+24t^2+8t+4$	4	4	52
4	2	$16t^4+16t^3+8t^2+8t+6$	6	6	54
Theoretical results		$16t^4+16t^3+8t^2+8t+6$	6	6	54
		$g_{n2}(t)-g_{n1}(t)$			
i, n		general formula	$t=-1$	$t=0$	$t=1$
0		10	10	10	10
1		$12t+6$	-6	6	18
2		$8t^2+4t-2$	2	-2	10
3		$8t^3+4t^2-4t-1$	-1	-1	7
4		0	0	0	0

(the latter equation resulting directly from Eq. (46)) as well as the values of b_i determined from Eq. (49) are presented in Table 2.

Finally, in Table 3, the approximate values $g_{n1}(t)$ and $g_{n2}(t)$ (for $n = 0, 1, \ldots, 10$) of $g(t)$ are displayed for $t = \pm 1$ in agreement with the developments of the previous section. We observe again the validity of Eq. (51) as well as the convergence of the numerical results for $g_{n1}(\pm 1)$ and $g_{n2}(\pm 1)$ to their theoretical values $g(\pm 1)$, which is sufficiently rapid. We can add that the numerical results for $g_{n2}(\pm 1)$ coincide with the corresponding numerical results already obtained in Ref. [21] directly for a crack problem although by a somewhat different numerical procedure.

Table 2

Values of the coefficients γ_i, b_i and e_i (for $i = 0, 1, \ldots, 10$) for the approximate solution of the airfoil equation (1) for $f(x) = (1-x)e^x$ by the methods of Section 3

i	γ_i	b_i	e_i
0	1.266065878	−0.565159104	0.000000000
1	0.565159104	0.293663764	1.130318208
2	0.135747670	0.204990065	0.542990679
3	0.022168425	0.055556794	0.133010550
4	0.002737120	0.009591165	0.021896962
5	0.000271463	0.001222384	0.002714631
6	0.000022489	0.000123737	0.000269864
7	0.000001599	0.000010398	0.000022389
8	0.000000100	0.000000747	0.000001594
9	0.000000005	0.000000047	0.000000099
10	0.000000000	0.000000003	0.000000005

As a final application, we consider the generalized airfoil equation (2) with kernel

$$\frac{1}{\pi(t-x)} + k(t,x) = \frac{1}{\beta H} \operatorname{cosech} \frac{\pi(t-x)}{\beta H}, \tag{58}$$

where βH is a constant. This particular kernel appears in the problem of the airfoil in a closed tunnel of height H in the case of steady flow [3]. The constant β is related to the Mach number M by the simple formula $\beta = \sqrt{1-M^2}$ [3]. From Eq. (58) we directly obtain

$$k(t,x) = \frac{1}{\beta H} \operatorname{cosech} \frac{\pi(t-x)}{\beta H} - \frac{1}{\pi(t-x)} \tag{59}$$

and because of Eqs. (22) and (26), we find that

$$\frac{1}{\pi(t-x)} + K(t,x) = \frac{1-t}{1-x}\,\frac{1}{\beta H}\left[\operatorname{cosech} \frac{\pi(t-x)}{\beta H} + \operatorname{cosech} \frac{\pi(1-t)}{\beta H}\right]. \tag{60}$$

Similarly, because of Eq. (25), from Eq. (59) we find that

$$-\frac{1}{\pi} + (1-t)k(t,1) = -\frac{1-t}{\beta H} \operatorname{cosech} \frac{\pi(1-t)}{\beta H}. \tag{61}$$

We have solved the Cauchy type singular integral equation (26) together with the corresponding condition (25) for

$$M = 0.85, \quad H = 15,\ 7.5\ \text{ and }\ 3.75 \quad \text{and} \quad f(x) = 1 \ \Rightarrow\ F(x) = 0 \tag{62}$$

because of Eq. (27) by using the classical Lobatto–Chebyshev method for the numerical solution of Cauchy type singular integral equations of the form (26) [15, 22]. In Table 4, we display the obtained numerical results for $n \geq 2$ nodes t_i in the Lobatto–Chebyshev quadrature rule (the number of collocation points x_k being $n-1$) for the quantities

$$f := \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, g_n(t)\,\mathrm{d}t \quad \text{and} \quad m := \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, t g_n(t)\,\mathrm{d}t. \tag{63}$$

Table 3

Numerical results for the approximate solutions $g_{n1}(t)$ and $g_{n2}(t)$ (for $t = \pm 1$) of the airfoil equation (1) for $f(x) = (1-x)e^x$ and $n = 0, 1, \ldots, 10$ by the methods of Section 3

n	$-g_{n1}(-1)$	$-g_{n2}(-1)$	$g_{n1}(1)$	$g_{n2}(1)$
0	0.56515910	0.00000000	−0.56515910	0.00000000
1	0.85882287	1.13031821	0.31583219	1.13031821
2	0.65383280	0.58732753	1.34078251	1.67330889
3	0.70938960	0.72033808	1.72968007	1.80631944
4	0.69979843	0.69844112	1.81600056	1.82821640
5	0.70102082	0.70115575	1.82944678	1.83093103
6	0.70089708	0.70088588	1.83105536	1.83120089
7	0.70090748	0.70090827	1.83121133	1.83122328
8	0.70090673	0.70090668	1.83122403	1.83122488
9	0.70090678	0.70090678	1.83122492	1.83122498
10	0.70090677	0.70090677	1.83122498	1.83122498
Theoretical values	0.70090677	0.70090677	1.83122498	1.83122498

These quantities are proportional to the lift and the moment, respectively. The above integrals (63) have been evaluated by the Lobatto–Chebyshev quadrature rule as well. The rapid convergence of the numerical results for the quantities f and m in Eqs. (63) for increasing values of the number of nodes n is clear from Table 4.

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[2] All the links (external links in blue) in this section were added by the author on 19 April 2018 for the online publication of this technical report. Moreover, final publication details were added in References [11, 14].

Table 4

Numerical results for the integrals f and m defined by Eqs. (63) obtained from the numerical solution of Eqs. (25) and (26) with the functions in these equations defined by Eqs. (58) and (62) by using the Lobatto–Chebyshev numerical method

n	$H = 15$	$H = 7.5$	$H = 3.75$
	f	f	f
2	−3.225015	−3.483283	−4.643760
3	−3.222625	−3.447664	−4.184312
4	−3.222624	−3.447543	−4.170577
5	−3.222624	−3.447543	−4.170226
6	−3.222624	−3.447543	−4.170217
≥ 7	−3.222624	−3.447543	−4.170216
	m	m	m
2	3.225015	3.483283	4.643760
3	1.590830	1.644045	1.792462
4	1.591199	1.649354	1.851573
5	1.591196	1.649236	1.847464
6	1.591196	1.649238	1.847599
7	1.591196	1.649238	1.847597
≥ 8	1.591196	1.649238	1.847596

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