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TECHNICAL REPORT No. TR-2019-Q9, 34 pages, November 14, 2019
deposited to *Nemertes* and appeared online also at *Nemertes*,
the institutional repository of the University of Patras

Interval computations in various direct and inverse applied mechanics problems related to quantifiers by using the method of quantifier elimination

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Abstract Quantifier elimination offers an interesting computational tool in many research areas including applied mechanics long ago. For example, quantifier elimination was recently applied to the computation of ranges of functions in problems of applied mechanics. Here we modify this approach by using the existential quantifier instead of the universal quantifier in the quantified formulae. This approach permits the reduction (by two) of the number of free variables. Yet, what is more important is that here we also extend this method based on quantifier elimination from the purely existential case to the mixed universal–existential case. The latter case is related to the classical interval tolerance and control problems so popular in interval analysis. Among the few implementations of quantifier elimination (in classical real analysis) in computer algebra systems again we selected the computer algebra system *Mathematica* for use in the present computations because it seems to offer the most efficient and user-friendly related implementation. Three applied mechanics problems are studied in detail: (i) a classical beam problem (beam fixed–simply-supported at its ends) under a uniform loading, (ii) a problem of a beam on a Winkler elastic foundation and (iii) the problem of free vibrations of the classical damped harmonic oscillator under critical damping. In these three problems, several quantified formulae were considered (of course, under appropriate assumptions) and the related QFFs (quantifier-free formulae) were easily derived. Moreover, the cases of (i) three interval variables and no parameter in the QFF, (ii) two interval variables and one parameter in the QFF and (iii) one interval variable and two parameters in the QFF were studied.

Keywords Intervals · Interval analysis · Interval variables/parameters · Range · Uncertainty · Uncertain variables/parameters · Beams · Beams on elastic foundation · Winkler foundation · Deflection · Vibration problems · Damped harmonic oscillator · Critical damping · Quantifiers · Universal quantifier · Existential quantifier · Quantified formulae · Quantified/free variables · Quantifier elimination · Quantifier-free formulae · Systems · Input · Output · Control problem · Tolerance problem · Direct problems · Inverse problems · Symbolic computations · *Mathematica*

1. Introduction

Computer algebra systems, which perform not only numerical but also symbolic computations, were proved very useful in several research areas including applied mechanics long ago. An interesting review of symbolic computations in structural engineering and applied mechanics in general was prepared by Pavlović [1] in 2003. On the other hand, quantifier elimination in elementary real algebra constitutes an interesting and rather recent computational approach in symbolic computations already implemented in few computer algebra systems. The most popular, efficient and general-purpose algorithm for quantifier elimination is CAD (cylindrical algebraic decomposition). This ingenious and very useful algorithm was devised and announced by Collins in 1973 at Carnegie Mellon University. The related fundamental publication is the paper by Collins [2] published in 1975. The book edited by Caviness and Johnson [3] and published (with some delay) in 1998 constitutes a very interesting collection of papers on quantifier elimination and CAD. Moreover, a bibliography on the applications of quantifier elimination in elementary real algebra was prepared by Ratschan [4] in 2012. On the other hand, a very large number of research results on CAD by several authors is available in the literature; see, e.g., the recent paper by Strzeboński [5].

The present results are based on the efficient and user-friendly implementation of quantifier elimination (based on CAD, but also incorporating several more quantifier-elimination algorithms) that is included in the computer algebra system *Mathematica* [6] and prepared by Strzeboński. In this author's opinion, this implementation is the most efficient related implementation and it will be continuously used for the derivation of the present results. The references to this implementation are the tutorial [7] and the related pages in the book by Trott [8, pp. 60–78] on symbolic computations.

Following many researchers in applied mechanics (since the sixties), this author has also been interested in the application of symbolic computations to applied mechanics during the last thirty years (see, e.g., Ref. [9]). During this period he used four computer algebra systems in his research: *Derive, Reduce, Maple* and *Mathematica*; see, e.g., Ref. [10] (published in 1992) concerning the use of *Mathematica* for the approximate solution of singular integral equations in crack problems.

In fact, this author feels that quantifier elimination (see, e.g., [2–4]) constitutes a very interesting possibility in symbolic computations in spite of its limitations and since 1994 he has been interested in its applications to several problems of applied mechanics; see, e.g., Refs. [11–26]. Furthermore, a related interesting recent paper, which directly used CAD and concerned optimal solutions to truss problems in structural mechanics, was prepared by Charalampakis and Chatzigiannelis [27].

On the other hand, the modern era of interval analysis began in 1959 with the famous related results by Moore and his collaborators; see, e.g., Refs. [28–30]. Previous results on interval analysis are due to several authors beginning in the antiquity with Archimedes for the computation of the number π. Additionally, an interesting recent bibliography on interval computations and reliable computing including 784 entries was prepared by Beebe, Kearfott and Kreinovich [31] in 2017.

The commands concerning interval arithmetic in *Mathematica* [6] are described by Keiper [32]. Moreover, a *Mathematica* package, the package `directed.m`, concerning directed interval arithmetic was prepared by Popova and Ullrich [33]. A second interesting *Mathematica* package, the package `IntervalComputations‘LinearSystems’`, devoted to the solution of parametric and nonparametric systems of linear equations with uncertainties was also prepared by Popova [34].

It is also well known that quantifiers and quantifier elimination are strongly related to interval analysis and this seems to be natural. Among the related results here we can make reference, e.g., to the results by Grandón and Neveu [35], Grandón and Goldsztejn [36] and Khanh and Ogawa [37].

At this point we can mention that the implementation of quantifier elimination in *Mathematica* [6] by Strzeboński was already successfully employed by Popova [38] and by Popova and Krämer [39] for the characterization of solution sets of parametric systems of linear algebraic equations. But, unfortunately, the derived results required too much CPU (central processing unit) time [38] or contained a very large number of logical expressions [39] in comparison with the same authors' own efficient methods for the same computational tasks for systems of linear equations.

Clearly, interval analysis proved to be an extremely useful tool in applied mechanics long ago. Among a very large number of related interesting publications see, e.g., the papers (in chronological order) by Dimarogonas [40], Qiu, Chen and Song [41], Qiu and Elishakoff [42], Kulpa, Pownuk and Skalna [43], McWilliam [44], Guo and Lü [45], Skalna [46], Popova, Iankov and Bonev [47], Shao and Su [48], Elishakoff and Ohsaki [49] (book), Wang and Qiu [50], Behera [51] (Ph.D. Thesis), Gabriele and Varano [52], Sofi, Muscolino and Elishakoff [53], Popova [54], Chakraverty, Hladík and Behera [55], Muscolino, Sofi and Giunta [56], Popova [57, 58], Skalna and Hladík [59], Faes and Moens [60, 61], Muscolino and Santoro [62], Sofi, Romeo, Barrera and Cocks [63], Dinh-Cong, Van Hoa and Nguyen-Thoi [64] and, very recently, Behera and Chakraverty [65]. The recent book by Skalna [66] on parametric interval systems also includes applications to applied mechanics.

In six recent technical reports [21–26], the author also combined quantifier elimination (using its implementation in *Mathematica* [6]) with interval analysis in the following problems: (i) the computation of ranges of functions appearing in problems of applied mechanics [21], (ii) the determination of ranges of values of stress concentration factors in plane elasticity (notch and hole problems) [22], (iii) similarly, for stress intensity factors at crack tips in fracture mechanics [23], (iv) the derivation of sharp enclosures of the real roots of the classical parametric quadratic equation with only one interval coefficient [24], (v) the determination of sharp bounds in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters [25] and, very recently, (vi) the derivation of symbolic intervals in simple problems of applied mechanics [26].

The purpose of this technical report is to modify and extend the recent results by the author [21] concerning the computation of ranges of functions of interest in applied mechanics. Here three of the applications in this reference are reconsidered (again by employing the method of quantifier elimination and *Mathematica* [6]), but now with the following two rather important modifications:

- The use of the existential quantifier $\exists$ instead of the universal quantifier $\forall$ in the quantified formula for the range of the function of interest. This modified approach permits the reduction of the number of free variables by two because the lower and upper bounds of this range do not appear as free variables in the quantified formula any more and this is very helpful because of the doubly-exponential computational complexity of quantifier elimination [67].
- Additionally and more importantly, the consideration of the general case of the computation of intervals related (i) either to the input x and/or (ii) the parameter p and/or (iii) the output $y = y(x, p)$ of the mechanical system under consideration instead of only the case of the output $y = y(x, p)$ having been originally studied in Ref. [21]. Moreover, either the universal quantifier $\forall$ or the existential quantifier $\exists$ or both these quantifiers are used in the quantified formulae. Naturally, the resulting intervals may be symbolic including one or two parameters among the three variables x, p and $y = y(x, p)$ in the mechanical system. (Of course, symbolic intervals are very well known long ago; see, e.g., the paper by Jaulin and Chabert [68].)

It is understood that the present applications to three applied mechanics problems (or here, equivalently and preferably, mechanical systems) more explicitly (i) a simple beam in Section 2, (ii) a beam on a Winkler elastic foundation in Section 3 and (iii) the classical damped harmonic oscillator under critical damping in Section 4 (all three also studied in Ref. [21]) are related to the general problem of a system with input x, parameter(s) p and output $y = y(x, p)$ having already and repeatedly studied in the related literature of interval analysis long ago. More explicitly, they also include the classical (i) interval tolerance problem and (ii) interval control problem, which are very well known in interval analysis. Among an extremely large number of related publications here we make reference to the papers by Shary [69–72], Beaumont and Philippe [73], Goldsztejn [74], Popova [75], Popova and Hladík [76] and Dymova, Sevastjanov, Pownuk, and Kreinovich [77].

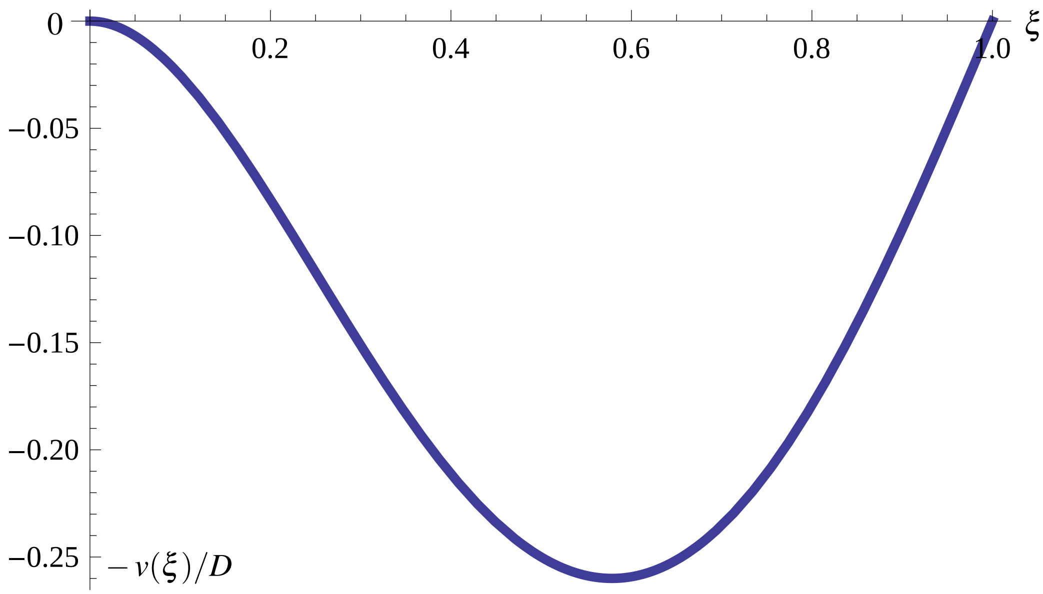

Fig. 1. The reduced deflection $v(\xi)/D$ (but with a minus sign, $-v(\xi)/D$) of the beam of Section 2.

2. A simple beam problem

2.1. The beam problem and the deflection of the beam

As a first application we consider the classical problem of applied mechanics concerning a simple straight beam of length L and flexural rigidity EI fixed at its left end $x = 0$ and simply-supported at its right end $x = L$ with $x \in [0, L]$. The beam is assumed loaded by a uniform (constant) loading p. Here we also assume that $p > 0$. Using the dimensionless length variable $\xi = x/L \in [0, 1]$, we find the deflection $v(\xi)$ of the beam, which has the form (see, e.g., Ref. [21, p. 8, Eqs. (30) and (31)])

$$v(\xi) = D(2\xi^4 - 5\xi^3 + 3\xi^2) = D\xi^2(2\xi^2 - 5\xi + 3) \quad \text{with} \quad \xi \in [0, 1] \quad \text{and} \quad D := \frac{pL^4}{48EI} > 0. \tag{1}$$

The graphical representation of the reduced deflection $v(\xi)/D$ (with a minus sign) is shown in Fig. 1.

This beam problem was recently studied in detail by the author [21, Subsection 3.1, pp. 8–10] with respect to the determination of the range of the deflection $v(\xi)$ of the present beam using the method of quantifier elimination. Here we will extend the results of Ref. [21] to additional related problems (beyond the range of $v(\xi)$ itself) using again the same method of quantifier elimination.

2.2. Determination of the range of the deflection of the beam

At first, the present beam problem concerns the computation of the range of the deflection $v(\xi)$ on the whole beam, i.e. with $\xi \in [0, 1]$. Of course, this is a direct problem. Our assumptions $\mathscr{A}_1$ have the form

$$\mathscr{A}_1 = 0 \le \xi \le 1 \wedge D > 0 \tag{2}$$

and they are denoted by the related symbol `ass1` in *Mathematica*. The related simple command is

```
ass1 = 0≤ξ≤1∧D>0                                                    [c1]
```

(or, alternatively, with `xi` instead of ξ). In this direct problem, we have the universally quantified formula

$$\forall \xi \text{ it holds true that } v_1 \le v(\xi) \le v_2 \text{ under the assumptions } \mathscr{A}_1 \tag{3}$$

with the universal quantifier $\forall$ (for all) for ξ and for the interval of the deflection $v(\xi)$ of the beam.

The related quantifier-elimination command, based on the `Reduce` command of *Mathematica*, is

```
Refine[Reduce[ForAll[ξ, ass1, v1≤vs≤v2], {v1,v2}, Reals], ass1]//Factor [c2]
```

also including the universal quantifier ∀ (for all). The auxiliary commands `Refine` and `Factor` have also been used for an improved appearance of the resulting QFF (quantifier-free formula). Moreover, the symbol `vs` in the above command [c2] simply denotes the deflection $v_s := v(\xi)$ of the beam in its closed-form formula (1) for $v(\xi)$ and it is not a free variable (but the variables v_1 and v_2 are free variables), i.e.

$$v_s := v(\xi) = D(2\xi^4 - 5\xi^3 + 3\xi^2) = D\xi^2(2\xi^2 - 5\xi + 3) = D\xi^2(\xi - 1)(2\xi - 3). \tag{4}$$

This formula was already determined (actually by using the `DSolve` command of *Mathematica* for the related simple boundary value problem for the present simple beam). The resulting QFF, which, evidently, holds true only under the assumptions $\mathscr{A}_1$ in Eq. (2), has the form

$$v_1 \le 0 \wedge v_2 \ge \frac{3(39 + 55\sqrt{33})D}{4096} \approx 0.259974D. \tag{5}$$

Therefore, the range of the deflection $v(\xi)$ of the present beam (with $\xi \in [0,1]$) is the interval

$$v(\xi) \in \left[0, \frac{3(39 + 55\sqrt{33})D}{4096}\right] \approx [0, 0.259974D] \tag{6}$$

or, completely equivalently, [21, p. 9, Eq. (35) and second of Eqs. (39)]

$$v(\xi) \in [0, HD] \quad \text{with} \quad H := \frac{3(39 + 55\sqrt{33})}{4096} \approx 0.259974 \tag{7}$$

because $D > 0$. The positivity of the overall parameter D is due to the fact that here for simplicity it was assumed that $p > 0$ for the uniform distributed loading p acting on the whole beam $\xi \in [0,1]$.

An alternative and preferable possibility not having been considered in Ref. [21] concerns the use of the existential quantifier ∃ (exists) instead of the universal quantifier ∀ (for all) in the quantified formula for the range of the deflection $v(\xi)$ of the beam. In this case, instead of the universally quantified formula (3) we have the modified and now existentially quantified analogous formula

$$\exists \xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_1 \tag{8}$$

with the symbol $v_s := v(\xi)$ defined again by Eq. (4). The serious computational advantage of the above existentially quantified formula (8) compared to the related universally quantified formula (3) simply consists in the fact that it avoids the two bounds v_1 and v_2 (the lower bound and the upper bound, respectively) of the actual range $[v_{s1}, v_{s2}]$ of the deflection $v_s := v(\xi)$ of the present simple beam. Therefore, the number of variables during quantifier elimination is now reduced by two.

In *Mathematica*, the quantifier-elimination command for the above existentially quantified formula (8), here based again on its `Reduce` command, has the form

```
Refine[Reduce[Exists[ξ, ass1, v==vs], v, Reals], ass1]//Factor        [c3]
```

The resulting QFF (quantifier-free formula) has the very simple form

$$0 \le v \le \frac{3(39 + 55\sqrt{33})D}{4096} \quad \text{and numerically} \quad 0 \le v \le 0.259974D. \tag{9}$$

Hence, we obtained again the same range (7), $[0, HD]$, for the deflection $v = v(\xi)$ of the beam, but now (i) in a somewhat simpler and more direct form than in the QFF (5) (where both bounds v_1 and v_2 appear), and, what is also important, (ii) in much less CPU (central processing unit) time: 0.016 s for the QFF (9) compared to 0.125 s for the QFF (5) in the computer used (with an Intel® Core™ i5 CPU at 3.20 GHz and with 4 GB of RAM under the operating system Windows 10). Therefore, it is now clear that for the determination of ranges of functions it is preferable to use an existentially quantified formula than a universally quantified formula having been used in Ref. [21].

Now we proceed to additional direct and inverse problems for the present beam. In all cases, we use the related quantified formulae (based on either the universal quantifier $\forall$ or on the existential quantifier $\exists$ or, simultaneously, on both these quantifiers $\forall$ and $\exists$), we perform the related quantifier elimination (always using the `Reduce` command of *Mathematica*) and we derive the related QFF (quantifier-free formula) providing us with the short reply `True` or `False` (this happens only in simple cases or with respect to `False` also in difficult cases) or, generally, in many cases, with the interval (or, if necessary, intervals) of interest in the problem under consideration for the present beam.

2.3. Problems with three interval variables and resulting QFFs without parameters: `True` *or* `False`

In the present beam problems, for the deflection $v = v(\xi)$ of the beam in Eqs. (1) we have three parameters: the dimensionless length variable $\xi \in [0,1]$ on the beam, the deflection v of the beam itself determined from the first of Eqs. (1) and the overall parameter D defined in the last of Eqs. (1) and for simplicity having been assumed positive, $D > 0$, because it was also assumed that $p > 0$.

As a first example in the present set of problems with three interval variables and without parameters in the resulting QFFs, we make the assumptions

$$\mathscr{A}_2 = 0 \le \xi \le 1 \land 1 \le D \le 2, \quad \text{i.e.} \quad \xi \in [0,1] \quad \text{and} \quad D \in [1,2] \tag{10}$$

denoted by the symbol `ass2` in *Mathematica*. Here we also assume that $v_s \in [1/3,1]$. The related existentially quantified formula has the form

$$\exists \xi \ \exists D \text{ such that } 1/3 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_2. \tag{11}$$

The related quantifier-elimination command in *Mathematica* (based again on its `Reduce` command) has the simple form

```
Reduce[Exists[{ξ, D}, ass2, 1/3 ≤ vs ≤ 1], Reals]        [c4]
```

The resulting QFF (quantifier-free formula) has the very simple form `True`. This fact does not seem to be unreasonable because in the present beam problem we defined three numerical intervals: (i) the interval $\xi \in [0,1]$ for the "input" ξ, (ii) the interval $D \in [1,2]$ for the overall parameter D and (iii) the interval $v_s \in [1/3,1]$ for the "output" v_s (i.e. the deflection of the beam). Therefore, the resulting QFF was really expected to include no interval being simply `True` or `False`. In the present problem, *Mathematica* found (after the appropriate existential quantifier elimination) that this QFF is equal to `True` and it is not equal to `False`. From the applied mechanics point of view this result means that under the above assumptions $\mathscr{A}_2$ in Eq. (10) (i.e. for the whole beam $\xi \in [0,1]$ and for the range of values $D \in [1,2]$ for the overall parameter D) there always exists at least one point $\xi \in [0,1]$ on the beam where its deflection $v = v(\xi)$ lies in the assumed interval $[1/3,1]$.

In quite a similar way, we used the following analogous quantifier-elimination command, but now with the universal quantifier $\forall$ instead of the existential quantifier $\exists$ in the above command [c4]:

```
Reduce[ForAll[{ξ,D}, ass2, 1/3 ≤ vs ≤ 1], Reals]        [c5]
```

with output, resulting QFF, now equal to `False` instead of `True` with the previous command [c4]. This is an expected result because now we assumed again that $\xi \in [0,1]$ in the assumptions $\mathscr{A}_2$ in Eq. (10) and, additionally, we demanded that $v_s \in [1/3,1]$ for all values of $\xi \in [0,1]$ (on the whole beam) whereas we surely know that $v(0) = v(1) = 0$. Of course, by using the similar command

```
Reduce[ForAll[{ξ,D}, ass2, 0 ≤ vs ≤ 1], Reals]        [c6]
```

but now with a different interval, i.e. the interval $v_s \in [0,1]$ instead of the interval $[1/3,1]$ in the command [c5], we obtain the QFF `True` instead of `False` previously. This change has happened because now the whole range $[0, 0.259974D]$ of the deflection $v(\xi)$ of the beam for all values of the overall parameter $D \in [1,2]$ is included in the interval $[0,1]$ assumed in the above command [c6].

Next, we proceed to two examples where both the universal quantifier $\forall$ and the existential quantifier $\exists$ are simultaneously present in the quantified formulae of these examples.

At first, we consider the quantified formula (now with both quantifiers $\forall$ and $\exists$)

$$\forall\xi\ \exists D \text{ such that } 1/3 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_2. \tag{12}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[ξ, ass2, Exists[D, ass2, 1/3 ≤ vs ≤ 1]], Reals], ass2]   [c7]
```

and the resulting QFF (quantifier-free formula) is simply `False`. But by changing again the interval of the deflection $v = v(\xi)$ from $[1/3, 1]$ to $[0, 1]$ using the completely analogous command

```
Refine[Reduce[ForAll[ξ, ass2, Exists[D, ass2, 0 ≤ vs ≤ 1]], Reals], ass2]   [c8]
```

we now get the QFF `True` instead of the QFF `False` previously with the command [c7], where the narrower interval $[1/3, 1]$ of the deflection $v = v(\xi)$ of the beam has been used.

Secondly we consider the inverse case where the universal quantifier $\forall$ (for all) concerns the overall parameter D and the existential quantifier $\exists$ (exists) concerns the dimensionless length variable ξ on the beam. In this case, we have the quantified formula (again with both quantifiers $\forall$ and $\exists$)

$$\forall D\ \exists\xi \text{ such that } 1/3 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_2 \tag{13}$$

instead of the quantified formula (12). The related quantifier-elimination command takes the form

```
Refine[Reduce[ForAll[D, ass2, Exists[ξ, ass2, 1/3 ≤ vs ≤ 1]], Reals], ass2]   [c9]
```

with output (QFF, quantifier-free formula) `False`. But by using again the interval $[0, 1]$ instead of the interval $[1/3, 1]$ for the deflection $v = v(\xi)$ of the beam through the slightly modified command

```
Refine[Reduce[ForAll[D, ass2, Exists[ξ, ass2, 0 ≤ vs ≤ 1]], Reals], ass2]   [c10]
```

we now obtain the QFF `True` completely analogously to the previous quite different case with respect to the two quantifiers $\forall$ and $\exists$ based on the quantifier-elimination command [c8].

2.4. Problems with two interval variables and resulting QFFs with one parameter

Now we proceed to the somewhat more interesting case where two of our three variables (i) the dimensionless length variable ξ, (ii) the overall parameter D and (iii) the deflection $v = v(\xi)$ of the present beam are interval variables whereas the remaining of these variables is an ordinary, non-interval parameter. (Evidently, the two inequality constraints $0 \le \xi \le 1$ and $D > 0$ remain valid in the present case and will be taken into account in our assumptions during quantifier eliminations.)

2.4.1. Interval variables ξ and D and parameter v

At first, we consider the simple case where we have the two interval variables ξ (the dimensionless length variable on the beam) and D (the overall parameter of the present beam problem) and one non-interval parameter (free variable), the parameter v, i.e. the deflection of the beam in Eq. (1) with its graphical representation shown in Fig. 1. The assumptions $\mathscr{A}_3$ made here have the form

$$\mathscr{A}_3 = 1/3 \le \xi \le 2/3 \wedge 1 \le D \le 2, \quad \text{i.e.} \quad \xi \in [1/3, 2/3] \subset [0, 1] \quad \text{and} \quad D \in [1, 2] \tag{14}$$

and they are denoted by the related symbol `ass3` in *Mathematica*.

We begin with the existentially quantified formula (with a double appearance of the existential quantifier $\exists$)

$$\exists\xi\ \exists D \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_3, \tag{15}$$

where, it is repeated, the symbol v_s is defined in Eqs. (4) and it denotes the deflection $v(\xi)$ of the beam in Eq. (1). The related quantifier-elimination command has the simple form

```
Reduce[Exists[{ξ,D}, ass3, v==vs], Reals]   [c11]
```

and the resulting QFF (quantifier-free formula) is

$$\frac{14}{81} \le v \le \frac{3(39+55\sqrt{33})}{2048} \quad \text{with numerical approximation} \quad 0.172840 \le v \le 0.519948. \tag{16}$$

Therefore, the interval of the deflection $v = v(\xi)$ (the range of this deflection) is

$$v \in \left[\frac{14}{81}, \frac{3(39+55\sqrt{33})}{2048}\right] \approx [0.172840, 0.519948]. \tag{17}$$

Next, we consider the following quantified formula (now with the appearance of both the universal quantifier $\forall$ and the existential quantifier $\exists$):

$$\forall\xi\ \exists D \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_3. \tag{18}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[ξ, ass3, Exists[D, ass3, v==vs]], Reals], ass3]        [c12]
```

The resulting QFF has the form

$$\frac{117+165\sqrt{33}}{4096} \le v \le \frac{28}{81} \quad \text{with numerical approximation} \quad 0.259974 \le v \le 0.345679. \tag{19}$$

Hence, the interval of the deflection $v = v(\xi)$ (the range of this deflection) is now

$$v \in \left[\frac{117+165\sqrt{33}}{4096}, \frac{28}{81}\right] \approx [0.259974, 0.345679]. \tag{20}$$

Here by comparing the two intervals (17) and (20), we observe that the second interval (20) is included in (is a subset of) the first interval (17). This is completely natural because the second interval (20) concerns all the points ξ in the part $[1/3, 2/3]$ of the beam (with $\forall\xi$ in the related quantified formula (18)) whereas the first interval (17) concerns at least one point of the same beam (with $\exists\xi$ in the related quantified formula (15) for the same part $[1/3, 2/3]$ of the beam). Therefore, the satisfaction of the first quantified formula (15) is clearly much easier than the satisfaction of the second quantified formula (18) and, therefore, it is completely reasonable that the interval (20) be included in (be a subset of) the interval (17) as has been already observed.

Next, we consider the quantified formula (again with the appearance of both the universal quantifier $\forall$ and the existential quantifier $\exists$, but now with the order of the variables ξ and D reversed)

$$\forall D\ \exists\xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_3. \tag{21}$$

The related quantifier-elimination command is similar to the command [c12] and has now the form

```
Refine[Reduce[ForAll[D, ass3, Exists[ξ, ass3, v==vs]], Reals], ass3]        [c13]
```

now with resulting QFF simply `False`.

Analogously, in the trivial case with the double appearance of the universal quantifier $\forall$ only, i.e. in the case with quantified formula

$$\forall\xi\ \forall D \text{ it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_3, \tag{22}$$

by using the related simple quantifier-elimination command

```
Reduce[ForAll[{ξ,D}, ass3, v==vs], Reals]        [c14]
```

we get again the QFF `False`, which is an expected result in this very difficult (from the physical point of view) case. This happens simply because it is impossible to have the same value v of the deflection v_s of the beam for all values of the interval dimensionless length variable $\xi \in [1/3, 2/3]$ on the beam and also the interval overall parameter $D \in [1, 2]$ in the assumptions $\mathscr{A}_3$ in Eq. (14).

2.4.2. Interval variables ξ and v and parameter D

Now we consider the case of the two interval variables ξ (the dimensionless length variable on the beam) and v (the deflection of the beam in Eq. (1) shown in dimensionless form, but with a minus sign, $-v(\xi)/D$, in Fig. 1) and parameter D (the overall parameter). Our assumptions $\mathscr{A}_4$ are

$$\mathscr{A}_4 = 1/3 \le \xi \le 2/3 \wedge 1/10 \le v \le 3/10, \text{ i.e. } \xi \in [1/3, 2/3] \subset [0,1] \text{ and } D \in [1/10, 3/10] \quad (23)$$

and they are denoted by the related symbol `ass4` in *Mathematica*.

We begin again with the existentially quantified formula

$$\exists \xi \ \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_4. \quad (24)$$

The related quantifier-elimination command is simply

```
Reduce[Exists[{ξ,v}, ass4, v==vs], Reals]                                [c15]
```

with resulting QFF

$$\frac{55\sqrt{33}-39}{720} \le D \le \frac{243}{140} \quad \text{with numerical approximation} \quad 0.384654 \le D \le 1.73571 \quad (25)$$

and related interval for the overall parameter D of the beam

$$D \in \left[\frac{55\sqrt{33}-39}{720}, \frac{243}{140}\right] \approx [0.384654, 1.73571]. \quad (26)$$

Now we proceed with the universally–existentially quantified formula

$$\forall \xi \ \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_4 \quad (27)$$

with the dimensionless length variable ξ on the beam universally quantified instead of existentially quantified in the previous quantified formula (24). The related quantifier-elimination command is

```
Refine[Reduce[ForAll[ξ, ass4, Exists[v, ass4, v==vs]], Reals], ass4]     [c16]
```

with resulting QFF

$$\frac{81}{140} \le D \le \frac{55\sqrt{33}-39}{240} \quad \text{with numerical approximation} \quad 0.578571 \le D \le 1.15396 \quad (28)$$

and related interval for the overall parameter D of the beam

$$D \in \left[\frac{81}{140}, \frac{55\sqrt{33}-39}{240}\right] \approx [0.578571, 1.15396]. \quad (29)$$

Evidently, the above interval $[0.578571, 1.15396]$ is a part (equivalently a subset) of the previous interval $[0.384654, 1.73571]$ for the same parameter D in the interval (26), which has been computed with $\exists \xi$ in the quantified formula (24) instead of $\forall \xi$ in the present quantified formula (27).

Next, we proceed with the 'inverse' universally–existentially quantified formula

$$\forall v \ \exists \xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_4 \quad (30)$$

now with the variable v for the deflection of the beam universally quantified instead of existentially quantified in the quantified formula (27) and, additionally, the inverse change with respect to the dimensionless length variable ξ. The related quantifier-elimination command is

```
Refine[Reduce[ForAll[v, ass4, Exists[ξ, ass4, v==vs]], Reals], ass4]     [c17]
```

In this case, the resulting QFF is simply `False` as is also, obviously, the case with the universally quantified formula

$$\forall \xi \ \forall v \text{ it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_4 \quad (31)$$

with related quantifier-elimination command

```
Reduce[ForAll[{ξ,v}, ass4, v==vs], Reals]
```
[c18]

with resulting QFF again simply `False` completely analogously to the previous sub-subsection for the similar quantified formula (22) there.

2.4.3. Interval variables D and v and parameter ξ

As a third case now we consider the case of the two interval variables D (the overall parameter of the present beam) and v (the deflection of the beam in Eq. (1) shown in dimensionless form, but with a minus sign, $-v(\xi)/D$, in Fig. 1) and parameter ξ (the dimensionless length variable on the beam with $\xi \in [0,1]$). Our assumptions $\mathscr{A}_5$ (denoted by the related symbol `ass5` in *Mathematica*) have now the form

$$\mathscr{A}_5 = 0 \le \xi \le 1 \wedge 1 \le D \le 3/2 \wedge 1/10 \le v \le 3/10 \tag{32}$$

with the first of them simply due to the physical constraint that $\xi = x/L \in [0,1]$ on the whole beam.

We begin again with the existentially quantified formula

$$\exists D\, \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_5 . \tag{33}$$

The related quantifier-elimination command is simply

```
Reduce[Exists[{D,v}, ass5, v==vs], Reals]
```
[c19]

with resulting QFF

$$s_{1,2} \le \xi \le s_{1,3} \quad \text{with numerical approximation} \quad 0.174546 \le \xi \le 0.932450, \tag{34}$$

where the symbols $s_{1,2}$ and $s_{1,3}$ denote the second and third real roots, respectively, of the quartic polynomial

$$p_1(s) := 30s^4 - 75s^3 + 45s^2 - 1. \tag{35}$$

Therefore, the related interval for the present parameter ξ (the dimensionless length variable on the beam) is

$$\xi \in [s_{1,2}, s_{1,3}] \approx [0.174546, 0.932450], \tag{36}$$

which, obviously, constitutes a part of the whole beam with $\xi \in [0,1]$. Moreover, in the present case, we directly observe that *Mathematica* provided us with a somewhat complicated exact QFF, which was displayed here in the simplified form (34). This happened because of the appearance of two roots, the roots $s_{1,2}$ and $s_{1,3}$ of the quartic polynomial $p_1(s)$ in Eq. (35), in the derived QFF (34).

Now we proceed again with the universally–existentially quantified formula

$$\forall D\, \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_5 \tag{37}$$

now with the overall parameter D universally quantified instead of existentially quantified as has been the case in the previous quantified formula (33). The related quantifier-elimination command is

```
Refine[Reduce[ForAll[D, ass5, Exists[v, ass5, v==vs]], Reals], ass5]
```
[c20]

with resulting QFF

$$s_{2,2} \le \xi \le s_{3,2} \vee s_{3,3} \le \xi \le s_{2,3} \tag{38}$$

with numerical approximation

$$0.224930 \le \xi \le 0.379020 \vee 0.768540 \le \xi \le 0.896942. \tag{39}$$

Here the symbols $s_{2,2}$ and $s_{2,3}$ denote the second and third real roots, respectively, of the quartic polynomial $p_2(s)$ and, analogously, the symbols $s_{3,2}$ and $s_{3,3}$ denote the second and third real roots, respectively, of the quartic polynomial $p_3(s)$. These two quartic polynomials $p_2(s)$ and $p_3(s)$ are defined as

$$p_2(s) := 20s^4 - 50s^3 + 30s^2 - 1, \quad p_3(s) := 10s^4 - 25s^3 + 15s^2 - 1. \tag{40}$$

Therefore, the related complete interval for the present parameter ξ is

$$\xi \in [s_{2,2}, s_{3,2}] \cup [s_{3,3}, s_{2,3}] \approx [0.224930, 0.379020] \cup [0.768540, 0.896942]. \tag{41}$$

With respect to this composite interval, at first we observe that it is the union of two separate intervals and not a single interval. This fact is justified since as is clear from Fig. 1, in the central part of the beam we cannot have $D = 3/2 = 1.5$ as is required by the second assumption $\mathscr{A}_5$ in Eq. (32) and also by the quantified formula (37) (with $\forall D$) and, simultaneously, the deflection v of the beam does not exceed the value $3/10 = 0.3$ as is required by the third assumption $\mathscr{A}_5$ in Eq. (32).

Moreover, again we also observe that the same composite interval (41) is a part (equivalently a subset) of the previous interval $[0.174546, 0.932450]$ of the same parameter ξ (the dimensionless length variable on the beam) in the interval (36), which has been obtained with $\exists D$ in the quantified formula (33) instead of $\forall D$ in the present and more difficult to be satisfied quantified formula (37).

Finally, we consider the 'inverse' universally–existentially quantified formula (with the rôles of the interval parameters D and v reversed in comparison with the previous quantified formula (37))

$$\forall v\, \exists D \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_5 \tag{42}$$

with related quantifier-elimination command

```
Refine[Reduce[ForAll[v, ass5, Exists[D, ass5, v==vs]], Reals], ass5]        [c21]
```

and resulting QFF simply `False`. The same is also the case with the completely (twice) universally quantified formula

$$\forall D\, \forall v \text{ it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_5 \tag{43}$$

with related quantifier-elimination command

```
Refine[Reduce[ForAll[{D,v}, ass5, v==vs], Reals], ass5]        [c22]
```

and resulting QFF again `False`. This QFF, `False`, is really expected in this particular and essentially trivial or, better, physically unacceptable case.

2.5. Problems with one interval variable and resulting QFFs with two parameters

Now we can proceed with the final case of quantifier-elimination problems, where only one of our three variables in the present beam problem, i.e. either (i) the dimensionless length variable ξ on the beam (with $\xi \in [0,1]$) or (ii) the overall parameter D of the beam (here with $D > 0$) or (iii) the deflection $v = v(\xi)$ of the beam, is an interval variable whereas the remaining two of the same three variables are ordinary, non-interval variables, i.e. simply parameters. Clearly, again the inequality constraints $0 \le \xi \le 1$ and $D > 0$ remain valid in the present case and they will be compatible with our assumptions $\mathscr{A}_6$, $\mathscr{A}_7$ or $\mathscr{A}_8$ to be made and actually used during quantifier elimination below.

2.5.1. Interval variable ξ and parameters D and v

We begin with the case where the dimensionless length variable ξ is assumed to be an interval variable here with $\xi \in [1/3, 2/3]$ or, equivalently, $1/3 \le \xi \le 2/3$. Here we also assume that $D > 0$ and $v > 0$. The related assumptions $\mathscr{A}_6$ (denoted by the related symbol `ass6` in *Mathematica*) are

$$\mathscr{A}_6 = 1/3 \le \xi \le 2/3 \wedge D > 0 \wedge v > 0. \tag{44}$$

At first, we consider the existentially quantified formula

$$\exists \xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_6 \tag{45}$$

with related quantifier-elimination command

```
Refine[Reduce[Exists[ξ, ass6, v==vs], D, Reals], ass6]//Factor   [c23]
```

and resulting two-parametric QFF (quantifier-free formula)

$$\frac{55\sqrt{33}-39}{72}v \le D \le \frac{81}{14}v \quad \text{and numerically} \quad 3.84654v \le D \le 5.78571v. \tag{46}$$

The related interval is

$$D \in \left[\frac{55\sqrt{33}-39}{72}v,\ \frac{81}{14}v\right] \approx [3.84654v, 5.78571v]. \tag{47}$$

The alternative but quite similar quantifier-elimination command

```
Refine[Reduce[Exists[ξ, ass6, v==vs],v,Reals], ass6]//Factor   [c24]
```

now permits the appearance of the resulting two-parametric present QFF solved with respect to the deflection v of the beam instead of the overall parameter D in the QFF (46). This completely equivalent QFF has the following form:

$$\frac{14}{81}D \le v \le \frac{3(39+55\sqrt{33})}{4096}D \quad \text{and numerically} \quad 0.172840D \le v \le 0.259974D. \tag{48}$$

The related interval is

$$v \in \left[\frac{14}{81}D,\ \frac{3(39+55\sqrt{33})}{4096}D\right] \approx [0.172840D, 0.259974D]. \tag{49}$$

Now, additionally, assuming that $1 \le D \le 2$ and using the quantifier-elimination command

```
Reduce[Exists[D, 1≤D≤2, QFF6], Reals]//Factor   [c25]
```

with the symbol QFF6 denoting the above QFF (48), we directly obtain the QFF (16) with only the deflection v as a parameter there. This constitutes a partial verification of the present QFF (48).

Finally, we can mention that by using the universal quantifier $\forall$ in lieu of the existential quantifier $\exists$ for the interval variable ξ in the above existentially quantified formula (45) (i.e. $\forall\xi$ it holds true that $v = v_s \ldots$), we directly get the simple and trivial QFF False as is really expected.

2.5.2. Interval variable D and parameters ξ and v

Now we continue with the case where the overall parameter D of the present beam problem is assumed to be an interval variable here with $D \in [1,2]$. We also make the assumptions $\mathscr{A}_7$ (denoted by the related symbol ass7 in *Mathematica* and including the previous assumption $D \in [1,2]$) that

$$\mathscr{A}_7 = 0 < \xi < 1 \wedge 1 \le D \le 2 \wedge v > 0. \tag{50}$$

At first, we consider the existentially quantified formula

$$\exists D \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_7 \tag{51}$$

with related quantifier-elimination command

```
Refine[Reduce[Exists[D, ass7, v==vs], Reals], ass7]//Factor   [c26]
```

and resulting two-parametric QFF (quantifier-free formula)

$$\xi^2(\xi-1)(2\xi-3) \le v \le 2\xi^2(\xi-1)(2\xi-3). \tag{52}$$

The related interval for the deflection $v = v(\xi)$ of the beam under the present assumptions $\mathscr{A}_7$ in Eq. (50) is

$$v \in [\xi^2(\xi-1)(2\xi-3), 2\xi^2(\xi-1)(2\xi-3)]. \tag{53}$$

This result is in complete agreement with Eq. (4), but here with $D \in [1,2]$ as was already assumed.

Finally, exactly as in the previous sub-subsection, here we can mention again that by using the universal quantifier $\forall$ instead of the existential quantifier $\exists$ now for the overall parameter D in the above existentially quantified formula (51) (i.e. $\forall D$ it holds true that $v = v_s \ldots$) after the related quantifier elimination we get the simple and trivial QFF `False` as is again really expected.

2.5.3. Interval variable v and parameters ξ and D

Next, we consider the third and final case where the deflection v of the beam is assumed to be an interval variable here with $v \in [1/10, 3/10]$ or, equivalently, $1/10 \le v \le 3/10$. Here we make the related assumptions $\mathscr{A}_8$ (denoted by the related symbol `ass8` in *Mathematica*)

$$\mathscr{A}_8 = 0 < \xi < 1 \wedge D > 0 \wedge 1/10 \le v \le 3/10. \tag{54}$$

Now we consider the existentially quantified formula

$$\exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_8 \tag{55}$$

with related quantifier-elimination command

```
Refine[Reduce[Exists[v, ass8, v==vs], Reals], ass8]//Factor        [c27]
```

and resulting two-parametric QFF (quantifier-free formula)

$$\frac{1}{10\xi^2(\xi-1)(2\xi-3)} \le D \le \frac{3}{10\xi^2(\xi-1)(2\xi-3)}. \tag{56}$$

Therefore, the range of the overall parameter D under the above assumptions $\mathscr{A}_8$ in Eq. (54) is

$$D \in \Big[\frac{1}{10\xi^2(\xi-1)(2\xi-3)}, \frac{3}{10\xi^2(\xi-1)(2\xi-3)}\Big]. \tag{57}$$

Naturally, alternatively, we can employ the similar quantifier-elimination command

```
Refine[Reduce[Exists[v, ass8, v==vs], ξ, Reals], ass8]//Factor        [c28]
```

with the resulting QFF now solved with respect to the dimensionless length variable ξ instead of the overall parameter D. By using the two quartic polynomials

$$p_4(s) := D(20s^4 - 50s^3 + 30s^2) - 1, \quad p_5(s) := D(20s^4 - 50s^3 + 30s^2) - 3, \tag{58}$$

we obtain (always with the help of *Mathematica*) the following QFF (quantifier-free formula):

$$\begin{aligned}&(D = 0.384654 \wedge \xi = s_{4,2})\\ &\vee (0.384654 < D \le 1.15396 \wedge s_{4,2} \le \xi \le s_{4,3})\\ &\vee [D > 1.15396 \wedge (s_{4,2} \le \xi \le s_{5,2} \vee s_{5,3} \le \xi \le s_{4,3})].\end{aligned} \tag{59}$$

In the above QFF (59), the two symbols $s_{4,2}$ and $s_{4,3}$ denote the second and the third real roots of the quartic polynomial $p_4(s)$ defined in the first of Eqs. (58) whereas the symbols $s_{5,2}$ and $s_{5,3}$ denote the corresponding roots of the quartic polynomial $p_5(s)$ defined in the second of Eqs. (58).

Of course, it is directly observed that the above QFF (59) is more difficult to use than the previous equivalent QFF (56). But, on the other hand, using the very simple command `QFF8/.D->1//N`, where the symbol `QFF8` denotes the above QFF (59), we get the numerical (essentially interval) QFF

$$0.224930 \le \xi \le 0.896942 \quad \text{and, equivalently,} \quad \xi \in [0.224930, 0.896942], \tag{60}$$

which is based on the second disjunctive term of the QFF (59). Analogously, using the similar and equally very simple command `QFF8/.D->2//N`, we get the slightly more complex numerical QFF

$$\begin{aligned}&0.147209 \le \xi \le 0.298200 \vee 0.838780 \le \xi \le 0.949629\\ &\qquad \text{and, equivalently,} \quad \xi \in [0.147209, 0.298200] \cup [0.838780, 0.949629]\end{aligned} \tag{61}$$

now based on the third disjunctive term of the above QFF (59) and consisting of two disjunctive terms or, equivalently, of the union of two distinct intervals. Of course, all the assumptions $\mathscr{A}_8$ in Eq. (54) should always hold true whenever we use the initial QFF (59) or the QFFs (60) and (61).

Finally, we can mention that by using the universal quantifier $\forall$ instead of the existential quantifier $\exists$ in the existentially quantified formula (55) (i.e. $\forall v$ it holds true that $v = v_s \ldots$), we obtain again the simple and trivial QFF `False` as is again really expected.

2.5.4. *Interval variable ξ in a symbolic interval and parameters D and v*

As a final application in the present beam problem, now we consider the interesting case where the dimensionless length variable ξ on the beam lies in a symbolic interval $[\xi_1, \xi_2]$, i.e. $\xi \in [\xi_1, \xi_2] \subseteq [0, 1]$, instead of a usual numerical interval, e.g. $\xi \in [1/3, 2/3]$ in the assumptions $\mathscr{A}_6$ in Eq. (44). Therefore, here we make the assumptions $\mathscr{A}_9$ (denoted by the related symbol `ass9` in *Mathematica*)

$$\mathscr{A}_9 = 0 \leq \xi_1 \leq \xi \leq \xi_2 \leq 1 \wedge D > 0 \wedge v > 0. \tag{62}$$

The present existentially quantified formula (here with interval variable the dimensionless length variable ξ) has the form

$$\exists \xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_9. \tag{63}$$

The related quantifier-elimination command (again based on the `Reduce` command) has the form

```
Refine[Reduce[Exists[ξ, ass9, v==vs], {ξ1,ξ2}, Reals], ass9]//Simplify   [c29]
```

The resulting QFF (quantifier-free formula) can be written in its final (and further simplified with the help of *Mathematica* and with the additional assumption $\xi_1 < \xi_2$ for a nontrivial interval) form

$$\left(\xi_1 \leq s_{6,2} \leq \xi_2 \wedge v \leq \frac{72D}{55\sqrt{33}-39}\right) \vee \left(s_{6,2} < \xi_1 \wedge \xi_1 \leq s_{6,3} \leq \xi_2 \wedge v < \frac{72D}{55\sqrt{33}-39}\right) \tag{64}$$

and in a further simplified form (since $72/(55\sqrt{33}-39) = 3(39+55\sqrt{33})/4096 \approx 0.259974$)

$$(\xi_1 \leq s_{6,2} \leq \xi_2 \wedge v \leq HD) \vee (s_{6,2} < \xi_1 \wedge \xi_1 \leq s_{6,3} \leq \xi_2 \wedge v < HD). \tag{65}$$

In this QFF, the QFF (65), the dimensionless symbol (the constant) H is the same symbol already defined in the right-hand side of Eq. (7), i.e. $H := 3(39+55\sqrt{33})/4096 \approx 0.259974$. Moreover, in the above QFF (64) or (65), the two symbols $s_{6,2}$ and $s_{6,3}$ denote the second and the third real roots, respectively, of the following quartic polynomial $p_6(s)$ with parameters the two parameters D and v:

$$p_6(s) := 2Ds^4 - 5Ds^3 + 3Ds^2 - v = Ds^2(2s^2 - 5s + 3) - v. \tag{66}$$

Clearly, alternative forms of the above QFF (64) or (65) can also easily be derived with *Mathematica* by changing the list of symbols `{ξ1,ξ2}` (concerning the order of variables) in the command [c29], e.g. by using the completely different list `{D,v}` instead of `{ξ1,ξ2}` in this command.

As a very simple but also rather interesting example of application of the above QFF (64) or (65), now we consider the case where $D = 1$ and $v = H$ with $H \approx 0.259974$ already defined in the right-hand side of Eq. (7). In this example, the above QFF (64) directly takes the extremely simple final form

$$\xi_1 \leq \frac{15-\sqrt{33}}{16} \leq \xi_2 \quad \text{and numerically} \quad \xi_1 \leq 0.578465 \leq \xi_2. \tag{67}$$

This result is explained on the basis of the fact that the reduced deflection $v(\xi)/D$ of the beam in Eq. (1) (with its graphical representation, but with a minus sign, in Fig. 1) takes its maximum value, which is simply $H \approx 0.259974$ (see the interval (7)), at the point $\xi_m = (15-\sqrt{33})/16 \approx 0.578465$ of the present beam. This fact is easily observed by computing (again with *Mathematica*) the roots of the first derivative $v'(\xi)/D$ of this reduced deflection $v(\xi)/D$. Hence, here with $D = 1$ and $v = H$ it is completely reasonable that the computed symbolic interval $[\xi_1, \xi_2]$ of ξ include the point ξ_m.

3. A problem concerning a beam on a Winkler elastic foundation

3.1. The beam problem on a Winkler elastic foundation and the reduced deflection of the beam

As a second application (again in applied mechanics) we consider the problem of a beam on a Winkler elastic foundation of modulus k_0 [78, p. 2]. More explicitly, we consider a cantilever beam of rectangular cross-section of width b, length L ($x \in [0,L]$) and flexural rigidity EI fixed (clamped) at its left end $x = 0$ and free at its right end $x = L$. The present beam on a Winkler elastic foundation is loaded by a compressive concentrated load P at its free end $x = L$. Here for convenience we use the dimensionless length variable $\xi := x/L \in [0,1]$. Then the deflection $y(\xi)$ of the beam has the form $y(\xi) = Q\,v(\xi)$ [78, p. 64], where Q is an appropriate positive overall parameter (constant) and the function $v(\xi)$ is a transcendental function. In the present beam problem, this function, $v(\xi)$, the reduced deflection of the beam on a Winkler elastic foundation, is seen to be given by [78, p. 64]

$$v(\xi) = \cosh\mu \sinh(\mu\xi)\cos[\mu(1-\xi)] - \cos\mu \sin(\mu\xi)\cosh[\mu(1-\xi)] \quad \text{with} \quad \xi \in [0,1]. \tag{68}$$

In this equation, the symbol μ ($\mu > 0$) denotes a dimensionless positive overall parameter (constant) of the present mechanical system, the system beam–Winkler elastic foundation, defined by

$$\mu := \lambda L \quad \text{with} \quad \lambda := \sqrt[4]{\frac{bk_0}{4EI}}. \tag{69}$$

Here this constant, μ, was already used for the simplification of Eq. (68). The second constant, λ, is an extremely well-known constant for a beam on a Winkler elastic foundation; see, e.g., [78, p. 4].

This classical problem of a beam on a Winkler elastic foundation was recently studied by the author [12, Section 6, pp. 69–70] with respect to the conditions of complete contact between the beam and the Winkler elastic foundation under the assumption of a tensionless Winkler foundation. This study was made by using the method of quantifier elimination based on its implementation in *Mathematica* [6] exactly as is also here the case. Of course, because quantifier elimination is generally possible only with polynomials and rational functions in the related quantified formulae, the following Taylor–Maclaurin series approximation $v_8(\xi)$ to the reduced deflection $v(\xi)$ of the beam in Eq. (68) (which is a transcendental function inappropriate for quantifier elimination):

$$v_8(\xi) = \frac{\mu^7}{315}\xi^7 - \frac{\mu^7}{45}\xi^6 + \left(\frac{\mu^7}{9} - \frac{2\mu^3}{3}\right)\xi^3 + \left(2\mu^3 - \frac{\mu^7}{15}\right)\xi^2 \tag{70}$$

was found (with the help of the `Series` command of *Mathematica* [6] for series expansions), see Ref. [21, Section 4, p. 11, Eq. (51)], and, next, it was actually used during quantifier elimination.

In the aforementioned Ref. [12, Section 6, pp. 69–70], using the method of quantifier elimination the author studied the continuous non-negativity of the above approximation $v_8(\xi)$ to the reduced deflection $v(\xi)$ of the beam (as well as of additional and improved related approximations) on the whole beam, i.e. for $\xi \in [0,1]$, and he found the related QFF (quantifier-free formula). This QFF can be written in the following simple final form [21, Section 4, pp. 11–12, Eqs. (52)–(54)]:

$$0 \le \mu \le \sqrt[4]{30} \approx 2.34034\,73193\,20715\,93845\,91410. \tag{71}$$

It is understood that the above QFF (71) is an approximate QFF simply because of the use of the approximation $v_8(\xi)$ in Eq. (70) to the exact reduced deflection $v(\xi)$ of the beam in Eq. (68).

On the other hand, in the aforementioned Ref. [21, Section 4, pp. 11–13], by using the method of quantifier elimination the author generalized the above result concerning only the non-negativity of the approximate reduced deflection $v_8(\xi)$ of the present beam on a Winkler elastic foundation to the determination of the range of this deflection $v_8(\xi)$ on the whole beam, i.e. with $\xi \in [0,1]$. In this way, he derived the two related QFFs and, finally, on the basis of these QFFs, the two related intervals [21, p. 13, Eq. (59)]. Naturally, the selection of the appropriate interval between these two intervals depends on the value of the overall parameter μ of the present system defined in Eq. (69).

In this section, analogously to the previous section for an ordinary beam, we will not restrict our attention to the direct problem, i.e. simply to the problem of determination of the range of the approximate reduced deflection $v_8(\xi)$ of the beam (here a cantilever beam or simply a cantilever) in Eq. (70). Here we intend to extend our interest in several direct and inverse problems related to this deflection $v_8(\xi)$ on the basis of the available three variables to be used as quantified and/or free variables: (i) the dimensionless length variable $\xi \in [0,1]$ on the present beam on a Winkler elastic foundation (the "input" of the mechanical system), (ii) the overall parameter (constant) μ in Eq. (69) of the present mechanical system, the system beam–Winkler elastic foundation, and (iii) the approximate reduced deflection $v_s := v_8(\xi)$ of the beam in Eq. (70) (the "output" of the same mechanical system). Moreover, we will use both quantifiers $\forall$ (for all), the universal quantifier, and $\exists$ (exists), the existential quantifier, exactly as has been already the case in the previous section for an ordinary beam. Naturally, it is clear that here because of the use of the approximation $v_8(\xi)$ in Eq. (70) to the reduced deflection $v(\xi)$ of the beam in Eq. (68), the derived QFFs (quantifier-free formulae) will be generally approximate QFFs of course with the possible exception of the simple cases `True` and `False`. In these two simple cases, the derived QFFs are frequently exact QFFs.

3.2. Problems with three interval variables and resulting QFFs without parameters: `True` *or* `False`

In the present set of beam problems, for the approximate reduced deflection $v_s = v_8(\xi)$ of the beam on a Winkler elastic foundation in Eq. (70) we have three interval variables (the interval variables ξ, μ and v_s, which are also quantified variables) mentioned in the last paragraph of the previous subsection and no parameters (no free variables).

Here for the interval variables ξ and μ we make the assumptions

$$\mathscr{A}_{10} = 0 \le \xi \le 1 \wedge 0 \le \mu \le 2.5, \quad \text{i.e.} \quad \xi \in [0,1] \quad \text{and} \quad \mu \in [0,2.5]. \tag{72}$$

These two assumptions $\mathscr{A}_{10}$ are denoted by the related symbol `ass10` in *Mathematica* (preferably in rational form) and they are defined in *Mathematica* by employing the simple command (leading to assumptions `ass10` appearing in rational form because of the use of the `Rationalize` command)

```
ass10 = 0≤ξ≤1∧0≤μ≤2.5//Rationalize
```
[c30]

For the approximate reduced deflection $v_s := v_8(\xi)$ of the beam here we also assume that $v_s \in [0,1]$.

At first, we consider the purely existential case with only the existential quantifier $\exists$ (exists). Under the above additional assumption that $v_s \in [0,1]$ the related existentially quantified formula has the form

$$\exists \xi \; \exists \mu \; \text{ such that } 0 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_{10}. \tag{73}$$

The related existential quantifier-elimination command in *Mathematica* (based again on its `Reduce` command exactly as in the previous section) has the simple form

```
Reduce[Exists[{ξ,μ},ass10,0≤vs≤1],Reals]
```
[c31]

Here the resulting QFF (quantifier-free formula) has the very simple form `True`. Exactly as in the previous section, this result does not seem to be unreasonable because in the present problem of a beam (here a cantilever) on a Winkler elastic foundation we defined three numerical intervals: (i) the interval $\xi \in [0,1]$ for the "input" ξ, (ii) the interval $\mu \in [0,2.5]$ for the overall parameter μ of the mechanical system beam–elastic foundation and (iii) the interval $v_s \in [0,1]$ for the "output" v_s. Hence, the resulting QFF was really expected to include no interval being simply `True` or `False`. In the present beam problem, *Mathematica* found that this QFF is equal to `True`. From the applied mechanics point of view this result means that under the above assumptions $\mathscr{A}_{10}$ in Eq. (72) (i.e. for the whole beam $\xi \in [0,1]$ and for the range of values $\mu \in [0,2.5]$ for the overall parameter μ of the system beam–Winkler elastic foundation) there always exists at least one point $\xi \in [0,1]$ on the beam where its approximate reduced deflection $v_s := v_8(\xi)$ lies in the assumed interval $[0,1]$.

In a completely similar manner, we used the following analogous quantifier-elimination command, but now with the universal quantifier $\forall$ instead of the existential quantifier $\exists$ in the above command [c31]:

```
Reduce[ForAll[{ξ,μ}, ass10, 0≤vs≤1], Reals]                                    [c32]
```

with output, resulting QFF, now equal to `False` instead of `True` with the previous command [c31]. This is an expected result because now we assumed again that $\xi \in [0,1]$ in the assumptions $\mathscr{A}_{10}$ in Eq. (72) and, additionally, we demanded that $v_s \in [0,1]$ for all values of $\xi \in [0,1]$ (on the whole beam) whereas we easily find with the help of *Mathematica* that for $\mu = 2.5 \in [0,2.5]$ in our assumptions $\mathscr{A}_{10}$ in Eq. (72) we have $v_s(0.1) = -0.0370142 \notin [0,1]$ and $v_s(1) = 36.3343 \notin [0,1]$ too.

Therefore, it seems now clear that by using the similar quantifier-elimination command

```
Reduce[ForAll[{ξ,μ}, ass10, -0.05≤vs≤40], Reals]                               [c33]
```

but now with a different, a broader interval, i.e. the interval $v_s \in [-0.05,40]$ instead of the original interval $[0,1]$ in the command [c32], we obtain the QFF `True` instead of `False` previously. This significant change has happened simply because now the whole range of the approximate reduced deflection $v_s := v_8(\xi)$ of the beam for all values of $\xi \in [0,1]$ and of the overall parameter $\mu \in [0,2.5]$ is included in the new interval $[-0.05,40]$, which was used in the above modified command [c33].

Next, we proceed to two simple examples where both the universal quantifier $\forall$ and also the existential quantifier $\exists$ are simultaneously present in the quantified formulae of these examples.

At first, we consider the quantified formula (now with both quantifiers $\forall$ and $\exists$)

$$\forall\xi\ \exists\mu \text{ such that } 0 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_{10}. \tag{74}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[ξ, ass10, Exists[μ, ass10, 0≤vs≤1]], Reals], ass10]      [c34]
```

and the resulting QFF (quantifier-free formula) is simply `True`.

Secondly we consider the inverse case where the universal quantifier $\forall$ (for all) concerns the overall parameter μ and the existential quantifier $\exists$ (exists) concerns the dimensionless length variable ξ on the beam. In this case, we have the quantified formula (again with both quantifiers $\forall$ and $\exists$)

$$\forall\mu\ \exists\xi \text{ such that } 0 \le v_s \le 1 \text{ under the assumptions } \mathscr{A}_{10} \tag{75}$$

instead of the quantified formula (74). The related quantifier-elimination command takes the form

```
Refine[Reduce[ForAll[μ, ass10, Exists[ξ, ass10, 0≤vs≤1]], Reals], ass10]      [c35]
```

with output (QFF, quantifier-free formula) again `True`. But, on the other hand, simply assuming now that $v_s \in [0.5,1]$ instead of $v_s \in [0,1]$ previously in the above command [c35], we obtain the completely different QFF `False` instead of `True` previously. This result, `False`, is naturally correct because $v_8(0) = 0 \notin [0.5,1]$ for any value of μ and this fact is clear from Eq. (70) for $v_s := v_8(\xi)$.

3.3. Problems with two interval variables and resulting QFFs with one parameter

Now we proceed to the somewhat more interesting case where two of our three variables (i) the dimensionless length variable ξ on the beam (here a cantilever) on a Winkler elastic foundation, (ii) the overall parameter μ of the present mechanical system beam–Winkler elastic foundation and (iii) the approximate reduced deflection $v = v_8(\xi)$ of the beam are interval variables whereas the remaining of these variables is an ordinary, non-interval parameter. (Obviously, the two inequality constraints $0 \le \xi \le 1$ and $\mu > 0$ remain valid in the present case and will be taken into account in our assumptions to be made during quantifier eliminations.) In the next three sub-subsections, we will separately consider the following three cases: (i) interval variables ξ and μ and parameter v, (ii) interval variables ξ and v and parameter μ and (iii) interval variables μ and v and parameter ξ.

3.3.1. Interval variables ξ and μ and parameter v

At first, we consider the simple case where we have the two interval variables ξ (the dimensionless length variable on the beam on a Winkler elastic foundation) and μ (the overall parameter of the present mechanical system) and one non-interval parameter (free variable), the parameter v, i.e. the approximate reduced deflection of the beam in Eq. (70) The assumptions $\mathscr{A}_{11}$ made here have the form

$$\mathscr{A}_{11} = 0 \le \xi \le 1 \wedge 0 < \mu \le 2.5, \quad \text{i.e.} \quad \xi \in [0,1] \quad \text{and} \quad \mu \in (0,2.5] \tag{76}$$

because $\mu > 0$. These assumptions are denoted by the related symbol `ass11` in *Mathematica*.

We begin again with the existentially quantified formula (with a double appearance of the existential quantifier $\exists$)

$$\exists\xi\ \exists\mu \ \text{such that } v = v_s \text{ under the assumptions } \mathscr{A}_{11}, \tag{77}$$

where, it is repeated, the symbol $v_s := v_8(\xi)$ denotes again the approximate reduced deflection $v_8(\xi)$ of the beam in Eq. (70). The related quantifier-elimination command has the simple form

```
Reduce[Exists[{ξ,μ}, ass11, v==vs], Reals]                                  [c36]
```

The resulting QFF (quantifier-free formula) is

$$s_{7,2} \le v \le \frac{36625}{1008} \tag{78}$$

and approximately

$$-0.03785\,00847\,14794\,50311\,5 \le v \le 36.33432\,53968\,25396\,825. \tag{79}$$

In the above QFF (78), the symbol $s_{7,2}$ denotes the second real root of the quintic polynomial

$$\begin{aligned} p_7(s) := {} & 77084825223168s^5 + 603907125999206400 0s^4 \\ & - 10758740407864565760 00s^3 + 6643358414723242800 0000s^2 \\ & - 1411563994004901347352000s - 5352304977867709145312 5. \end{aligned} \tag{80}$$

Therefore, the interval of the approximate reduced deflection $v = v_8(\xi)$ of the present beam on a Winkler elastic foundation (here the range of this approximate deflection for $\mu \in (0,2.5]$) is

$$v \in \left[s_{7,2}, \frac{36625}{1008}\right] \approx [-0.03785\,00847\,14794\,50311\,5, 36.33432\,53968\,25396\,825]. \tag{81}$$

Now we consider the following quantified formula (here with the appearance of both the universal quantifier $\forall$ and the existential quantifier $\exists$):

$$\forall\xi\ \exists\mu \ \text{such that } v = v_s \text{ under the assumptions } \mathscr{A}_{11}. \tag{82}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[ξ, ass11, Exists[μ, ass11, v==vs]], Reals], ass11]   [c37]
```

The resulting QFF is now simply `False`.

Next, we consider the analogous quantified formula (but now with the order of the two quantified variables ξ and μ reversed)

$$\forall\mu\ \exists\xi \ \text{such that } v = v_s \text{ under the assumptions } \mathscr{A}_{11}. \tag{83}$$

The related quantifier-elimination command is similar to the command [c37] and has now the form

```
Refine[Reduce[ForAll[μ, ass11, Exists[ξ, ass11, v==vs]], Reals], ass11]   [c38]
```

In this particular case, again under the assumptions $\mathscr{A}_{11}$ the resulting QFF has the trivial form $v = 0$.

Analogously, finally, in the trivial and somewhat strange case concerning the double appearance of the universal quantifier $\forall$ only (for both interval variables ξ and μ), i.e. in the case with quantified formula

$$\forall\xi\ \forall\mu \ \text{it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_{11}, \tag{84}$$

using the related quantifier-elimination command we get again the trivial but expected QFF `False`.

3.3.2. Interval variables ξ and v and parameter μ

Now we proceed with the case where we have the two interval variables ξ (the dimensionless length variable on the beam on a Winkler elastic foundation) and v (the approximate reduced deflection of the beam in Eq. (70)). On the other hand, μ (the overall parameter of the present mechanical system) is a non-interval parameter (a free variable). The related assumptions $\mathscr{A}_{12}$ have the form

$$\mathscr{A}_{12} = 0 \le \xi \le 1 \land 0 \le v \le 1 \land 0 < \mu \le 2.5, \quad \text{i.e.} \quad \xi \in [0,1], \quad v \in [0,1] \quad \text{and} \quad \mu \in (0, 2.5]. \tag{85}$$

These assumptions are denoted by the related symbol `ass12` in *Mathematica*.

We begin again with the existentially quantified formula (with a double appearance of the existential quantifier $\exists$)

$$\exists \xi \ \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{12}. \tag{86}$$

Here, it is repeated, the symbol $v_s := v_8(\xi)$ denotes again the approximate reduced deflection $v_8(\xi)$ of the present beam in Eq. (70). The related quantifier-elimination command has the form

```
Refine[Reduce[Exists[{ξ,v}, ass12, v==vs], Reals], ass12]    [c39]
```

In this purely existential case, the resulting QFF (quantifier-free formula) has the simple form `True`.

Now we consider the following quantified formula here with the appearance of both the universal quantifier $\forall$ (for the quantified variable ξ) and the existential quantifier $\exists$ (for the quantified variable v):

$$\forall \xi \ \exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{12}. \tag{87}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[ξ, ass12, Exists[v, ass12, v==vs]], Reals], ass12]    [c40]
```

The resulting QFF (with the single parameter μ and under the assumptions $\mathscr{A}_{12}$) has now the form

$$\mu \le s_{8,1}, \quad \text{numerically} \quad \mu \le 0.90472\,78950\,49830\,83246 \tag{88}$$

and in an interval form (since μ is a positive parameter because of Eqs. (69))

$$\mu \in (0, s_{8,1}] \approx (0, 0.90472\,78950\,49830\,83246]. \tag{89}$$

In this QFF the symbol $s_{8,1}$ denotes the first (and single) real root of the seventh-degree polynomial

$$p_8(s) := 8s^7 + 420s^3 - 315. \tag{90}$$

Next, we consider the analogous quantified formula (but now with the order of the two quantified variables ξ and v reversed)

$$\forall v \ \exists \xi \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{12}. \tag{91}$$

The related quantifier-elimination command is similar to the command [c40] and has now the form

```
Refine[Reduce[ForAll[v, ass12, Exists[ξ, ass12, v==vs]], Reals], ass12]    [c41]
```

In this particular case, again under the assumptions $\mathscr{A}_{12}$ in Eq. (85), the resulting QFF takes the form

$$\mu \ge s_{8,1}, \quad \text{numerically} \quad \mu \ge 0.90472\,78950\,49830\,83246 \tag{92}$$

and in an interval form, which is analogous to the previous interval form (89),

$$\mu \in [s_{8,1}, 5/2] \approx [0.90472\,78950\,49830\,83246, 2.5] \tag{93}$$

because the positive parameter $\mu \in (0, 2.5]$. Here the symbol $s_{8,1}$ denotes again the first (and actually single) real root of the seventh-degree polynomial $p_8(s)$ already defined in Eq. (90).

Finally, in the trivial and somewhat strange case concerning the double appearance of the universal quantifier $\forall$ only (for both interval variables ξ and v), i.e. in the case with quantified formula

$$\forall \xi \ \forall v \text{ it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_{12}, \tag{94}$$

using the related quantifier-elimination command we get again the trivial but expected QFF `False`.

3.3.3. Interval variables μ and v and parameter ξ

Here we consider the third case, i.e. the case where we have the two interval variables μ (the overall parameter of the present mechanical system beam–Winkler elastic foundation) and v (the approximate reduced deflection of the beam in Eq. (70)) whereas ξ (the dimensionless length variable on the beam on a Winkler elastic foundation) is now a non-interval parameter, but, of course, with $\xi \in [0,1]$. The assumptions $\mathscr{A}_{13}$ made here have the form

$$\mathscr{A}_{13} = 0 < \xi \leq 1 \wedge 0 \leq v \leq 1 \wedge 0 < \mu \leq 2.5, \quad \text{i.e.} \quad \xi \in (0,1], \quad v \in [0,1] \quad \text{and} \quad \mu \in (0, 2.5] \tag{95}$$

since $\mu > 0$. These assumptions are denoted by the related symbol `ass13` in *Mathematica*.

We begin again with the related existentially quantified formula (here with a double appearance of the existential quantifier $\exists$)

$$\exists \mu \ \exists v \ \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{13}, \tag{96}$$

where, it is repeated, the symbol $v_s := v_8(\xi)$ denotes again the approximate reduced deflection $v_8(\xi)$ of the beam in Eq. (70). The related quantifier-elimination command has the form

```
Refine[Reduce[Exists[{μ,v}, ass13, v==vs], Reals], ass13]     [c42]
```

The resulting QFF (quantifier-free formula) has the simple and more or less expected form `True`.

Now we consider the following quantified formula (here with the appearance of both the universal quantifier $\forall$ and the existential quantifier $\exists$):

$$\forall \mu \ \exists v \ \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{13}. \tag{97}$$

The related quantifier-elimination command has the form

```
Refine[Reduce[ForAll[μ, ass13, Exists[v, ass13, v==vs]], Reals], ass13]     [c43]
```

The resulting QFF has the form

$$s_{9,1} \leq \xi \leq s_{10,2} \tag{98}$$

and numerically

$$0.16463\,07446\,51136\,72662 \leq \xi \leq 0.30324\,72347\,18251\,66070. \tag{99}$$

In the above QFF (98), the symbol $s_{9,1}$ denotes the first real root (out of three real roots) of the quintic polynomial

$$p_9(s) := 125s^5 - 875s^4 + 3703s - 609. \tag{100}$$

Moreover, in the same QFF, the symbol $s_{10,2}$ denotes the second real root (again out of three real roots) of the fifteenth-degree polynomial

$$\begin{aligned} p_{10}(s) := {} & 36861599s^{15} - 774093579s^{14} + 6966943053s^{13} - 34835656457s^{12} \\ & + 104509187895s^{11} - 188113311267s^{10} + 188102521173s^9 \\ & - 80614156113s^8 - 8823675s^7 + 72354135s^6 - 77295393s^5 \\ & + 22235661s^4 - 102942875s^3 + 185297175s^2 - 111178305s + 22235661. \end{aligned} \tag{101}$$

Therefore, in the present case of the universally–existentially quantified formula (97), the resulting interval for the parameter ξ (the dimensionless length variable on the beam) has the form

$$\xi \in [s_{9,1}, s_{10,2}] \approx [0.16463\,07446\,51136\,72662, 0.30324\,72347\,18251\,66070]. \tag{102}$$

Next, we consider the analogous quantified formula (but now with the order of the two quantified interval variables μ and v in the present beam problem reversed)

$$\forall v \ \exists \mu \ \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{13}. \tag{103}$$

The related quantifier-elimination command is similar to the command [c43]. In this particular case, again under the assumptions $\mathscr{A}_{13}$ in Eq. (95) the resulting QFF has the trivial form `False`.

Analogously, finally, in the trivial case of the double appearance of the universal quantifier $\forall$ only (for both quantified interval variables μ and v), i.e. in the case with quantified formula

$$\forall\mu\ \forall v\ \text{ it holds true that } v = v_s \text{ under the assumptions } \mathscr{A}_{13}, \tag{104}$$

using the related quantifier-elimination command we get again the trivial but expected QFF `False`.

3.4. Problems with one interval variable and resulting QFFs with two parameters

Now we can proceed with the final case of quantifier-elimination problems, where only one of our three variables in the present beam problem, i.e. either (i) the dimensionless length variable ξ on the present beam on a Winkler elastic foundation (with $\xi \in [0,1]$) or (ii) the overall parameter μ of the mechanical system beam–Winkler elastic foundation (with $\mu > 0$) or (iii) the approximate reduced deflection $v = v_8(\xi)$ of the beam, is an interval variable whereas the remaining two of the same three variables are ordinary, non-interval variables, i.e. simply parameters. Clearly, again the inequality constraints $0 \le \xi \le 1$ and $\mu > 0$ remain valid in the present case and will be used below.

3.4.1. Interval variable ξ and parameters μ and v

We begin with the case where the dimensionless length variable ξ is assumed to be an interval variable here with $\xi \in [0,1]$ or, equivalently, $0 \le \xi \le 1$. This assumption concerns the whole beam. Here we additionally assume that $\mu \in (0, 2.5]$ or, equivalently, that $0 < \mu \le 2.5$. Now the related assumptions $\mathscr{A}_{14}$ (denoted by the symbol `ass14` in *Mathematica* and used in a rational form) are

$$\mathscr{A}_{14} = 0 \le \xi \le 1 \wedge 0 < \mu \le 2.5. \tag{105}$$

Here we consider the existentially quantified formula

$$\exists\xi\ \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{14} \tag{106}$$

with related quantifier-elimination command

```
Refine[Reduce[Exists[ξ, ass14, v==vs], Reals], ass14]//Factor    [c44]
```

The resulting approximate two-parametric QFF (quantifier-free formula) has the form

$$(\mu \le s_{11,2} \wedge 0 \le v \le v_{m1}) \vee (s_{11,2} < \mu \wedge s_{12,2} \le v \le v_{m1}). \tag{107}$$

In the above QFF, the symbol $s_{11,2}$ denotes the second real root of the quartic polynomial

$$p_{11}(s) := s^4 - 30 \tag{108}$$

(this root already used in the right-hand side of Eq. (71)) whereas the symbol $s_{12,2}$ denotes the second real root of the parametric (with parameter the positive overall parameter μ) quintic polynomial

$$\begin{aligned}
p_{12}(s) := {} & 114865340625s^5 + (14532106522500\mu^7 + 8270304525000\mu^3)s^4 \\
& + (-5358340512000\mu^{14} + 43128106560000\mu^{10} - 73513818000000\mu^6 \\
& - 63011844000000\mu^2)s^3 + (827355513600\mu^{21} - 20931934464000\mu^{17} \\
& + 266210038080000\mu^{13} - 1809940204800000\mu^9 - 504094752000000\mu^5)s^2 \\
& + (-24991948800\mu^{28} - 295132723200\mu^{24} + 47169247104000\mu^{20} \\
& - 963274394880000\mu^{16} + 6450126681600000\mu^{12} - 11686768128000000\mu^8 \\
& - 9876142080000000\mu^4 - 1763596800000000)s - 102449152\mu^{35} \\
& + 8297103360\mu^{31} - 82027008000\mu^{27} - 11069872128000\mu^{23} \\
& + 442806497280000\mu^{19} - 6458160844800000\mu^{15} + 32753258496000000\mu^{11} \\
& + 25642137600000000\mu^7 + 4702924800000000\mu^3.
\end{aligned} \tag{109}$$

Finally, in the same QFF (107), the symbol v_{m1} simply denotes the quantity

$$v_{m1} := \frac{4}{315}\mu^3(2\mu^4 + 105). \tag{110}$$

Of course, the related two symbolic intervals for the approximate reduced deflection v of the beam (with parameter the positive overall parameter μ of the present mechanical system beam–Winkler elastic foundation) are

$$v \in [0, v_{m1}] \quad \text{valid for } \mu \in (0, s_{11,2}] \quad \text{and} \quad v \in [s_{12,2}, v_{m1}] \quad \text{valid for } \mu \in (s_{11,2}, 2.5]. \tag{111}$$

We observe that the above results are in agreement with the corresponding results having been obtained in Ref. [21, Section 4, pp. 11–13] by a different quantifier-elimination method based on the universal quantifier $\forall$ and two bounds v_1 and v_2 for v instead of the existential quantifier $\exists$ here.

Moreover, naturally, the above approximate two-parametric QFF (107) can be directly used for concrete values of the overall parameter μ. For example, for $\mu = 2$ this QFF takes the simple form

$$0 \le v \le 13.91746\,03174\,60317\,460, \quad \text{i.e.} \quad v \in [0, 13.91746\,03174\,60317\,460]. \tag{112}$$

This result is more or less obvious because $v_8(0) = 0$ (at the left, fixed end $\xi = 0$ of the beam) and for $\mu = 2$ we find that $v_8(1) = 13.91746\,03174\,60317\,460$ (at the right, free end $\xi = 1$ of the beam).

3.4.2. Interval variable μ and parameters ξ and v

We continue with the second case, where the overall parameter μ (with $\mu > 0$) is an interval variable whereas the dimensionless length variable ξ (with $\xi \in [0,1]$ on the beam) and the approximate reduced deflection v of the beam are non-interval variables, i.e. simply parameters. The related assumptions $\mathscr{A}_{15}$ (denoted by the symbol `ass15` in *Mathematica* and used here again in a rational form obtained with the help of the command `Rationalize`) are

$$\mathscr{A}_{15} = 0 \le \xi \le 1 \land 0 < \mu \le 2.5 \land v > 0. \tag{113}$$

In the present case, we consider the existentially quantified formula

$$\exists \mu \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{15} \tag{114}$$

with related quantifier-elimination command

```
Refine[Reduce[Exists[μ, ass15, v==vs], {ξ,v}, Reals], ass15]        [c45]
```

The resulting two-parametric QFF (quantifier-free formula) with parameters ξ and v has the form

$$(0 < \xi \le s_{13,1} \land v \le s_{14,2}) \lor (\xi > s_{13,1} \land v \le v_{\xi 1}), \tag{115}$$

where $v_{\xi 1}$ denotes the quantity (seventh-degree polynomial with respect to the parameter ξ)

$$v_{\xi 1} = \frac{1}{8064}(15625\xi^7 - 109375\xi^6 + 462875\xi^3 - 76125\xi^2). \tag{116}$$

Moreover, in the above QFF (115), the symbol $s_{13,1}$ denotes the first real root (out of three real roots) of the quintic polynomial

$$p_{13}(s) := 125s^5 - 875s^4 + 4087s - 1761 \quad \text{with} \quad s_{13,1} \approx 0.43828\,37209\,39213\,15201. \tag{117}$$

On the other hand, in the same QFF, the symbol $s_{14,2}$ denotes the second real root of the parametric quartic polynomial (with parameter the dimensionless length variable ξ on the beam)

$$\begin{aligned} p_{14}(s) := (&2401\xi^{15} - 50421\xi^{14} + 352947\xi^{13} - 823543\xi^{12} + 252105\xi^{11} - 3680733\xi^{10} \\ &+ 14470827\xi^9 - 7411887\xi^8 + 8823675\xi^7 - 72354135\xi^6 + 77295393\xi^5 \\ &- 22235661\xi^4 + 102942875\xi^3 - 185297175\xi^2 + 111178305\xi - 22235661)s^4 \\ &- 36864000\xi^{15} + 774144000\xi^{14} - 6967296000\xi^{13} + 34836480000\xi^{12} \\ &- 104509440000\xi^{11} + 188116992000\xi^{10} - 188116992000\xi^9 + 80621568000\xi^8. \end{aligned} \tag{118}$$

Therefore, under the assumptions $\mathscr{A}_{15}$ in Eq. (113) the resulting two symbolic intervals for the approximate reduced deflection v of the present beam on a Winkler elastic foundation are

$$v \in (0, s_{14,2}] \quad \text{valid for} \quad \xi \in (0, s_{13,1}] \quad \text{and} \quad v \in (0, v_{\xi 1}] \quad \text{valid for} \quad \xi \in (s_{13,1}, 1]. \tag{119}$$

3.4.3. Interval variable v and parameters ξ and μ

Finally, we consider the third case, where the approximate reduced deflection v of the beam is an interval variable whereas the dimensionless length variable ξ (with $\xi \in [0,1]$ on the beam) and the overall parameter μ (with $\mu > 0$) of the mechanical system are non-interval variables, i.e. simply parameters. The related assumptions $\mathscr{A}_{16}$ (denoted by the symbol `ass16` in *Mathematica*) are

$$\mathscr{A}_{16} = 0 \leq \xi \leq 1 \wedge 0 < v \leq 1 \wedge \mu > 0 \tag{120}$$

since, it is repeated, the overall parameter μ is a positive constant because of Eqs. (69).

In the present case, we consider the existentially quantified formula

$$\exists v \text{ such that } v = v_s \text{ under the assumptions } \mathscr{A}_{16} \tag{121}$$

with related quantifier-elimination command (here with the order (μ, ξ) of the parameters ξ and μ)

```
Refine[Reduce[Exists[v, ass16, v==vs], {μ,ξ}, Reals], ass16]//N    [c46]
```

The resulting two-parametric approximate and numerical QFF (quantifier-free formula) with parameters ξ and μ has the form

$$\begin{aligned}&(\mu \leq 0.904728 \wedge \xi > 0) \vee (0.904728 < \mu \leq 1.96496 \wedge 0 < \xi \leq s_{15,3})\\ &\quad \vee (1.96496 < \mu \leq 2.34035 \wedge 0 < \xi \leq s_{15,1}) \vee (\mu > 2.34035 \wedge s_{16,1} < \xi \leq s_{15,1}).\end{aligned} \tag{122}$$

In the above QFF (122), the symbols $s_{15,1}$ and $s_{15,3}$ denote the first and the third real roots, respectively, of the seventh-degree parametric polynomial (here with parameter μ)

$$p_{15}(s) := \mu^7 s^7 - 7\mu^7 s^6 + (35\mu^7 - 210\mu^3)s^3 + (-21\mu^7 + 630\mu^3)s^2 - 315. \tag{123}$$

Moreover, in the same QFF, the symbol $s_{16,1}$ denotes the first real root of the quintic parametric polynomial (again with parameter μ)

$$p_{16}(s) := \mu^4 s^5 - 7\mu^4 s^4 + (35\mu^4 - 210)s - 21\mu^4 + 630. \tag{124}$$

Naturally, we can also work with the order (ξ, μ) instead of (μ, ξ) in the quantifier-elimination command [c46]. The resulting approximate and numerical QFF has the essentially equivalent form

$$\begin{aligned}&(0 < \xi \leq 0.303247 \wedge \mu < s_{17,2})\\ &\quad \vee [0.303247 < \xi < 0.628385 \wedge (\mu \leq s_{18,2} \vee s_{18,3} \leq \mu < s_{17,2})]\\ &\quad \vee (0.628385 \leq \xi \wedge \mu \leq s_{18,1})\end{aligned} \tag{125}$$

of course valid only under the assumptions $\mathscr{A}_{16}$ in Eq. (120). In the above QFF (125), the symbol $s_{17,2}$ denotes the second real root of the quartic parametric polynomial (now with parameter ξ)

$$p_{17}(s) := (\xi^5 - 7\xi^4 + 35\xi - 21)s^4 - 210\xi + 630. \tag{126}$$

Analogously, in the same QFF, the symbols $s_{18,1}$, $s_{18,2}$ and $s_{18,3}$ denote the first, the second and the third real roots, respectively, of the seventh-degree parametric polynomial (again with parameter ξ)

$$p_{18}(s) := (\xi^7 - 7\xi^6 + 35\xi^3 - 21\xi^2)s^7 + (-210\xi^3 + 630\xi^2)s^3 - 315. \tag{127}$$

Evidently, from the QFFs (122) and (125) we can directly obtain the related intervals for the parameters ξ and μ, respectively. Of course, these intervals are symbolic intervals (with parameters μ and ξ, respectively) and they depend on the values of the same parameters μ and ξ, respectively.

4. Free vibrations of the classical damped harmonic oscillator under critical damping

4.1. The problem of the damped harmonic oscillator under critical damping

As a third and final application now we consider the problem of free vibrations of the classical damped harmonic oscillator here under critical damping (i.e. with damping ratio $\zeta = 1$). By using the method of quantifier elimination this classical problem was already studied by the author in Ref. [21, Section 6, pp. 18–21], but only with respect to the range of the displacement $u = u(t)$ (with t denoting the time) of the mass of the harmonic oscillator from its equilibrium position $u = 0$.

The approximate solution $u_7(\tau)$ of the related well-known second-order differential equation (by using a polynomial minimax approximation of degree $n = 7$ to the exponential function $e^{-\tau}$ on the interval $\tau \in [0,5]$) based on the exact solution $u(\tau)$ of this differential equation (including the exponential function $e^{-\tau}$) has the following reduced dimensionless form [21, p. 19, Eq. (93)]:

$$\begin{aligned} u_7(\tau) = & -0.0000136748c\tau^8 + (0.000372942c - 0.0000136748)\tau^7 \\ & + (0.000372942 - 0.00452513c)\tau^6 + (0.0324597c - 0.00452513)\tau^5 \\ & + (0.0324597 - 0.153187c)\tau^4 + (0.489270c - 0.153187)\tau^3 \\ & + (0.489270 - 0.996320c)\tau^2 + (0.999747c - 0.996320)\tau + 0.999747. \end{aligned} \tag{128}$$

In this equation, the symbol τ is defined as $\tau := \omega_0 t$ (with $\tau \geq 0$) with the symbol ω_0 denoting the natural frequency of vibrations of the related oscillator without damping ($\zeta = 0$) and t denotes the time (with $t \geq 0$). Moreover, the symbol c denotes a parameter depending on the initial conditions.

In Ref. [21, Section 6, pp. 18–21], we restricted our attention to the determination of the range of the function $u_7(\tau)$, the approximate reduced displacement of the mass of the oscillator, using the method of quantifier elimination, two related bounds u_1 (lower bound) and u_2 (upper bound) and the universal quantifier $\forall$ (for all) for the interval variable $\tau \in [0,5]$ and the parameter (free variable) c.

On the contrary, here, analogously to the previous two sections, for the determination of the range of the same function $u_7(\tau)$ by using the method of quantifier elimination we will not use the bounds u_1 and u_2 of the function $u_7(\tau)$ and we will use the existential quantifier $\exists$ (exists) instead of the universal quantifier $\forall$ (for all). This approach significantly reduces the computational effort and time in quantifier elimination because of the reduced (by two) number of variables. Moreover, here, again analogously to the previous two sections, we will also consider few additional quantifier-elimination problems that are related (i) to three interval variables (τ, c and u) and no parameters, (ii) to two interval variables and one parameter and (iii) to one interval variable and two parameters.

4.2. Problems with three interval variables and resulting QFFs without parameters: `True` *or* `False`

We begin with the simplest case of three interval variables (τ, c and u) and no parameters. Here we make the assumptions

$$\mathscr{A}_{17} = 0 \leq \tau \leq 5 \wedge 0 \leq c \leq 1 \tag{129}$$

denoted by the related symbol `ass17` in *Mathematica*.

For the existentially quantified formula

$$\exists\tau\,\exists c \text{ such that } u_{\min} \leq u_s \leq u_{\max} \text{ under the assumptions } \mathscr{A}_{17}, \tag{130}$$

where the symbol u_s denotes the minimax approximation $u_7(\tau)$ in Eq. (128) to the reduced displacement of the mass of the oscillator, i.e. $u_s := u_7(\tau)$, and under the additional assumptions $u_{\min} = 0$ and $u_{\max} = 2$, i.e. $u_s \in [0,2]$, by using the related simple quantifier-elimination command

```
Reduce[Exists[{τ,c},ass17,0≤us≤2],Reals]                                    [c47]
```

we directly find the QFF (quantifier-free formula) `True`. On the contrary, by using the interval $u_s \in [1,2] \subset [0,2]$ instead of the interval $u_s \in [0,2]$ previously, we find a different QFF: `False`.

Completely analogously, for the related universally–existentially quantified formula

$$\forall\tau\,\exists c \text{ such that } u_{\min} \le u_s \le u_{\max} \text{ under the assumptions } \mathscr{A}_{17} \tag{131}$$

for the first interval $u_s \in [0,2]$ by using the quantifier-elimination command

```
Refine[Reduce[ForAll[τ, ass17, Exists[c, ass17, 0 ≤ us ≤ 2]], Reals], ass17]   [c48]
```

we obtain the QFF `True`. On the contrary, for the second and narrower interval $u_s \in [1,2] \subset [0,2]$ we find the different QFF `False` exactly as previously in the purely existential case (130).

Similarly, for the 'inverse' universally–existentially quantified formula

$$\forall c\,\exists\tau \text{ such that } u_{\min} \le u_s \le u_{\max} \text{ under the assumptions } \mathscr{A}_{17} \tag{132}$$

we get again the QFFs `True` and `False` for the two intervals $u_s \in [0,2]$ and $u_s \in [1,2]$, respectively.

Completely analogous is the final case concerning the universally quantified formula

$$\forall\tau\,\forall c \text{ it holds true that } u_{\min} \le u_s \le u_{\max} \text{ under the assumptions } \mathscr{A}_{17}\,. \tag{133}$$

Even in this purely universal case, in the present damped harmonic oscillator problem, we obtain again the QFFs `True` and `False` for the same two intervals $u_s \in [0,2]$ and $u_s \in [1,2]$, respectively.

4.3. Problems with two interval variables and resulting QFFs with one parameter

Now we proceed with the more interesting case of two interval variables and QFFs (quantifier-free formulae) with one parameter.

At first, we consider the case of the two interval variables τ and c and parameter u. In this case, under the assumptions

$$\mathscr{A}_{18} = 1 \le \tau \le 4 \wedge 1 \le c \le 10 \tag{134}$$

by using the existentially quantified formula

$$\exists\tau\,\exists c \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{18} \tag{135}$$

and the related command of *Mathematica*, we obtain the approximate and here numerical too QFF

$$0.0916008 \le u \le 4.04584, \quad \text{i.e.} \quad u \in [0.0916008, 4.04584]. \tag{136}$$

On the other hand, for the related universally–existentially quantified formula

$$\forall\tau\,\exists c \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{18} \tag{137}$$

using the related command of *Mathematica*, we obtain the approximate and here numerical too QFF

$$0.735608 \le u \le 0.751127, \quad \text{i.e.} \quad u \in [0.735608, 0.751127]. \tag{138}$$

Evidently, this much more narrow interval (since $[0.735608, 0.751127] \subset [0.0916008, 4.04584]$) is simply due to the fact that now for the interval variable τ we used the universal quantifier $\forall$ ($\forall\tau$) in the formula (137) instead of the existential quantifier $\exists$ ($\exists\tau$) previously in the formula (135).

From the physical (evidently not from the computational) point of view even more difficult is the 'inverse' case with respect to the two interval variables τ and c with universally–existentially quantified formula

$$\forall c\,\exists\tau \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{18}\,. \tag{139}$$

In this case, we obtain the simple QFF `False`. This means that the above quantified formula does not hold true (of course, here always under the interval assumptions $\mathscr{A}_{18}$ in Eq. (134)) for any value of the parameter u, i.e. the approximate reduced displacement of the mass of the present oscillator.

Similarly, in the physically even more difficult case of the universally quantified formula

$$\forall\tau\,\forall c \text{ it holds true that } u = u_s \text{ under the assumptions } \mathscr{A}_{18}, \tag{140}$$

we get again the QFF `False` as is now naturally expected because of the already obtained previous result `False` for the physically easier to hold true quantified formula (139).

Next, we consider the case of the two interval variables τ and u now with parameter c. Here we make the assumptions

$$\mathscr{A}_{19} = 1 \le \tau \le 4 \wedge 0 \le u \le 2 \wedge c \ge 0 \tag{141}$$

denoted by the related symbol `ass19` in *Mathematica*.

In this case, at first, we consider the related existentially quantified formula

$$\exists\tau\,\exists u \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{19}, \tag{142}$$

where the symbol u_s denotes again the eighth-degree polynomial minimax approximation $u_7(\tau)$ in Eq. (128) to the reduced displacement $u(\tau)$ of the mass of the present oscillator ($u_7(\tau) \approx u(\tau)$), i.e. $u_s := u_7(\tau)$. Then by using the related simple quantifier-elimination command

```
Reduce[Exists[{τ,u}, ass19, u==us], Reals]
```
[c49]

we obtain the related approximate numerical QFF (quantifier-free formula) for the parameter c

$$0 \le c \le 27.0423, \quad \text{i.e.} \quad c \in [0, 27.0423]. \tag{143}$$

Completely analogously, for the related universally–existentially quantified formula

$$\forall\tau\,\exists u \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{19} \tag{144}$$

we obtain the approximate numerical QFF

$$c \le 4.43768, \quad \text{i.e.} \quad c \in [0, 4.43768] \tag{145}$$

because of the present assumptions $\mathscr{A}_{19}$ in Eq. (141). Evidently, this much more narrow interval (because $[0, 4.43768] \subset [0, 27.0423]$) is again obviously simply due to the fact that now for the interval variable τ we used the universal quantifier $\forall$ ($\forall\tau$) in the quantified formula (144) instead of the existential quantifier $\exists$ ($\exists\tau$) having been used previously in the quantified formula (142).

Finally, we can also consider the case of the two interval variables c and u and of parameter τ. In this case, here we make the assumptions

$$\mathscr{A}_{20} = 1 \le c \le 10 \wedge 0 \le u \le 4 \wedge 0 \le \tau \le 5 \tag{146}$$

since $\tau \in [0,5]$ because this was the interval having been used in the original polynomial minimax approximation having led to Eq. (128) for the approximation $u_7(\tau)$ to the displacement of the mass of the oscillator. The above assumptions are denoted by the related symbol `ass20` in *Mathematica*.

For the related existentially quantified formula

$$\exists c\,\exists u \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{20} \tag{147}$$

we can proceed to quantifier elimination by using the related command

```
Refine[Reduce[Exists[{c,u}, ass20, u==us], Reals], ass20]
```
[c50]

The resulting QFF has the very simple form `True`.

Somewhat more interesting seems to be the related universally–existentially quantified formula

$$\forall c\,\exists u \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{20}. \tag{148}$$

This happens since by using the related quantifier-elimination command

```
Refine[Reduce[ForAll[c, ass20, Exists[u, ass20, u==us]], Reals], ass20]    [c51]
```

we obtain the following QFF essentially consisting of the union of two disjoint (separate) intervals:

$$\tau \le 0.730204 \vee \tau \ge 1.09003, \quad \text{i.e.} \ \ \tau \in [0, 0.730204] \cup [1.09003, 5] \quad \text{since} \ \ \tau \in [0,5]. \tag{149}$$

Finally, we can mention that both QFFs corresponding to the two remaining quantified formulae

$$\forall u \, \exists c \ \text{such that} \ u = u_s \ \text{under the assumptions} \ \mathscr{A}_{20} \tag{150}$$

and

$$\forall c \, \forall u \ \text{it holds true that} \ u = u_s \ \text{under the assumptions} \ \mathscr{A}_{20} \tag{151}$$

were directly found to be simply `False` under the above assumptions $\mathscr{A}_{20}$ in Eq. (146).

4.4. Problems with one interval variable and resulting QFFs with two parameters

In the present problem of free vibrations of a critically damped harmonic oscillator, now we proceed to the case of only one interval variable. Then the resulting QFFs will have two parameters.

At first, we assume that the variable τ is the interval variable and we make the assumptions

$$\mathscr{A}_{21} = 1 \le \tau \le 4 \wedge c \ge \frac{1}{10} \tag{152}$$

denoted by the related symbol `ass21` in *Mathematica*. We also assume the validity of the related existentially quantified formula. This formula has the form

$$\exists \tau \ \text{such that} \ u = u_s \ \text{under the assumptions} \ \mathscr{A}_{21}. \tag{153}$$

By using the related quantifier-elimination command

```
Refine[Reduce[Exists[τ, ass21, u==us], {c,u}, Reals], ass21]//Factor    [c52]
```

we obtain the following two-parametric QFF (quantifier-free formula):

$$0.0732807(c + 0.25) \le u \le 0.367804(c + 1). \tag{154}$$

Alternatively, by using the similar quantifier-elimination command

```
Refine[Reduce[Exists[τ, ass21, u==us], {u,c}, Reals], ass21]//Expand    [c53]
```

(now with the order of the two parameters c and u reversed) or even the slightly simpler command

```
Refine[Reduce[Exists[τ, ass21, u==us], Reals], ass21]//Expand    [c54]
```

we obtain the equivalent, but now more complicated, two-parametric QFF

$$\begin{aligned}(u = 0.0256482 \wedge c = 0.1) \vee (0.0256482 < u \le 0.404584 \wedge c \le 13.6462u - 0.25)\\ \vee (u > 0.404584 \wedge 2.71884u - 1 \le c \le 13.6462u - 0.25)\end{aligned} \tag{155}$$

of course under the continuous validity of the above assumptions $\mathscr{A}_{21}$ in Eq. (152).

Remark: At this point we can remark that these assumptions are not explicitly displayed in this QFF, evidently, in a negated form and only as far as the free variables, the parameters (here only the parameter c, second assumption $\mathscr{A}_{21}$) are concerned. This happens simply because of the use of the `Refine` command with respect to these assumptions $\mathscr{A}_{21}$. This approach, based on the use of the `Refine` command, frequently permits the appearance of the resulting QFFs in simpler forms, but, of course, assumed valid only under the related assumptions, here the assumptions $\mathscr{A}_{21}$ in Eq. (152).

On the other hand, by considering the related universally quantified formula

$$\forall \tau \ \text{it holds true that} \ u = u_s \ \text{under the assumptions} \ \mathscr{A}_{21} \tag{156}$$

and using the related quantifier-elimination command (now with `ForAll` instead of `Exists` for τ)

```
Refine[Reduce[ForAll[τ, ass21, u==us], Reals], ass21]    [c55]
```

we get the simple non-parametric QFF `False`.

As a second application with one interval variable and two parameters now we consider the case of the interval variable c (here with $c \in [1,2]$) with parameters τ and u. We make the assumptions

$$\mathscr{A}_{22} = 1 \leq c \leq 2 \wedge 0 \leq \tau \leq 5 \tag{157}$$

(since $\tau \in [0,5]$) denoted by the related symbol `ass22` in *Mathematica*. We also assume the validity of the related existentially quantified formula. This formula has now the form

$$\exists c \text{ such that } u = u_s \text{ under the assumptions } \mathscr{A}_{22}. \tag{158}$$

By using the related quantifier-elimination command

```
Refine[Reduce[Exists[c, ass22, u==us], Reals], ass22]//Expand    [c56]
```

we obtain the related two-parametric QFF (quantifier-free formula) now with parameters τ and u. This QFF can easily be written (continuously with the help of *Mathematica*) in its final numerical form

$$(\tau = 0 \wedge u = 0.999747) \vee [\tau > 0 \wedge p_{19}(\tau) \leq u \leq p_{20}(\tau)]. \tag{159}$$

In this QFF, the symbol $p_{19}(\tau)$ denotes the non-parametric eighth-degree polynomial

$$\begin{aligned} p_{19}(\tau) := & -0.0000136748\tau^8 + 0.000359268\tau^7 - 0.00415219\tau^6 + 0.0279346\tau^5 \\ & -0.120727\tau^4 + 0.336083\tau^3 - 0.507050\tau^2 + 0.00342664\tau + 0.999747 \end{aligned} \tag{160}$$

whereas, analogously, the symbol $p_{20}(\tau)$ denotes the non-parametric eighth-degree polynomial

$$\begin{aligned} p_{20}(\tau) := & -0.0000273496\tau^8 + 0.000732210\tau^7 - 0.00867733\tau^6 + 0.0603943\tau^5 \\ & -0.273913\tau^4 + 0.825353\tau^3 - 1.50337\tau^2 + 1.00317\tau + 0.999747. \end{aligned} \tag{161}$$

With the help of *Mathematica* it can easily be verified (as was actually suspected) that

$$p_{19}(\tau) = u_7(\tau) \quad \text{for} \quad c = 1 \quad \text{and} \quad p_{20}(\tau) = u_7(\tau) \quad \text{for} \quad c = 2, \tag{162}$$

where the polynomial minimax approximation $u_7(\tau)$ is displayed in Eq. (128) and here we have $u_s := u_7(\tau)$. The above result simply means that here with $c \in [1,2]$ the range of the approximate reduced displacement $u = u_7(\tau)$ of the mass of the present oscillator under critical damping ($\zeta = 1$)

$$u \in [p_{19}(\tau), p_{20}(\tau)] \quad \text{for} \quad \tau \in [0, 5] \tag{163}$$

has as endpoints $p_{19}(\tau)$ and $p_{20}(\tau)$ the values of this function $u = u_7(\tau)$ corresponding to the two endpoints $c = 1$ and $c = 2$ of the interval $c \in [1,2]$ of the interval variable c in the assumptions $\mathscr{A}_{22}$.

Finally, we consider the related universally quantified formula (with respect to the same quantified interval variable c)

$$\forall c \text{ it holds true that } u = u_s \text{ under the assumptions } \mathscr{A}_{22}. \tag{164}$$

Then using the related quantifier-elimination command (of course now with the quantifier `ForAll` for the quantified variable c instead of the quantifier `Exists` previously in the command [c56]), i.e.

```
Refine[Reduce[ForAll[c, ass22, u==us], Reals], ass22]    [c57]
```

we get the rather trivial approximate numerical QFF

$$\tau = 0 \wedge u = 0.999747. \tag{165}$$

This QFF seems to be obvious by taking into consideration Eq. (128) for the minimax approximation $u_7(\tau)$ to the reduced displacement $u(\tau)$ of the mass of the present damped harmonic oscillator.

5. Conclusions–discussion

From the present results it is directly concluded that the well-known computational method of quantifier elimination (here based on its efficient and user-friendly implementation in the computer algebra system *Mathematica* [6]) can be efficiently used for the computation of intervals concerning quantities appearing both in direct and in inverse applied mechanics problems under uncertainty conditions. Moreover, in the case of simple mechanical systems (such as the simple beam in Section 2, the beam on a Winkler elastic foundation in Section 3 and the critically damped harmonic oscillator in Section 4), where we generally have an input x, a mechanical parameter p and an output $y = y(x, p)$, the present approach is applicable to all possible combinations of interval variables (e.g. x, p and/or y) and of simple parameters, i.e. free variables (e.g. again x, p and/or y), with the parameters generally appearing in the resulting QFF (quantifier-free formula). Moreover, here both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) can appear in the quantified formula, on which quantifier elimination is performed for the derivation of the related sought QFF.

The present results constitute one more possibility of application of quantifier elimination to applied mechanics. More explicitly, in comparison with the results of Ref. [21] concerning only the computation of ranges of functions, the present results are much more general since they concern not only the output $y = y(x, p)$ of the mechanical system, but they uniformly concern the input x, the mechanical parameter p and the output $y = y(x, p)$ with each one of these three variables x, p and y having the possibility to be assumed either as an interval variable (evidently not appearing in the resulting QFF) or as a free variable (parameter), which generally appears in the resulting QFF.

Moreover, again in comparison with the results of Ref. [21], in the particular case of the ranges of functions exclusively considered in this reference, the present approach uses the existential quantifier $\exists$ instead of the universal quantifier $\forall$ having been used in this reference and, in this way, it avoids both the lower bounds u_1 and the upper bounds u_2 of the interval quantity u of interest. From the computational point of view this change of the quantifier used is particularly helpful since, in this way, the number of free variables is reduced by two and this permits the significant reduction of the computational time and, moreover, the study of more difficult quantifier-elimination problems.

Of course, it is always understood that quantifier elimination (here in real analysis) presents some difficulties during its application to applied mechanics and additional problems: (i) At first, there are only very few implementations of quantifier elimination in computer algebra systems, the most powerful of which seems to be that available in *Mathematica* [6] and having been used here. (ii) Secondly, quantifier elimination is generally applicable only to polynomial and rational functions and, therefore, transcendental functions (such as the exponential, the trigonometric and the hyperbolic functions) cannot be directly used in the quantified formula. In order to avoid this difficulty here in Section 3 we used a Taylor–Maclaurin series approximation to the reduced deflection $v(\xi)$ of the beam on a Winkler elastic foundation and in Section 4 we used a polynomial minimax approximation to the exponential function $e^{-\tau}$ appearing in the reduced displacement $u(\tau)$ of the mass of a critically damped harmonic oscillator. Naturally, the second approach is generally preferable to the first one since it takes into account the whole interval of the independent variable x in the function $y = y(x, p)$ under consideration. (iii) Finally and most importantly, unfortunately, quantifier elimination for real variables has a doubly-exponential computational complexity [67] and this constitutes a significant obstacle to its wide application especially for a large total number of variables (e.g. more than four or five). Additionally, clearly, the degrees of the polynomials in the quantified formulae should not be large. For example, here we used a quartic polynomial in Section 2, a seventh-degree polynomial in Section 3 and an eighth-degree polynomial in Section 4.

Naturally, in spite of the above undoubted restrictions, the previous successful applications of quantifier elimination to both direct and inverse applied mechanics problems illustrate its practical usefulness and it is really hoped that its actual use in applied mechanics will increase in the future.

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