Professor Emeritus Nikolaos I. Ioakimidis, former professor at the

DIVISION OF APPLIED MATHEMATICS AND MECHANICS
DEPARTMENT OF ENGINEERING SCIENCES
SCHOOL OF ENGINEERING, UNIVERSITY OF PATRAS

TECHNICAL REPORT No. TR-2020-Q11, 30 pages, February 16, 2020
deposited to *Nemertes* and appeared online also at *Nemertes*,
the institutional repository of the University of Patras

Determination of intervals in systems of parametric interval linear equilibrium equations in applied mechanics with the method of quantifier elimination

Nikolaos I. Ioakimidis

School of Engineering, University of Patras, GR-265 04 Patras, Greece
e-mail: n.ioakimidis@upatras.gr

Abstract The method of quantifier elimination is an interesting computational tool in computer algebra with many practical applications including problems of applied mechanics. Recently, this method was used in applied mechanics problems with uncertain parameters varying in known intervals (interval parameters) including systems of parametric interval linear equilibrium equations, direct and inverse problems and the computation of resultants of interval forces. Here the case of systems of parametric interval linear equilibrium equations is further considered by using related quantified formulae including not only the existential quantifier (as is the case with the united solution set of such a system), but also both the universal and the existential quantifiers in the quantified formula (a more general case) with respect to the parameters of the problem including the external loads applied to the mechanical system. Two problems of applied mechanics related to systems of parametric interval linear equilibrium equations are studied in detail: (i) the problem of a simply-supported truss with two external loads recently studied under uncertainty (interval) conditions by E. D. Popova and (ii) the problem of a clamped bar with a gap subjected to a concentrated load recently studied again under uncertainty (interval) conditions by E. D. Popova and I. Elishakoff. Here, in both these problems, by using the method of quantifier elimination both (i) complete solution sets for the unknown quantities (here mainly reactions) and (ii) separate intervals for each unknown quantity are computed on the basis of related quantified formulae. The present results are compared to the results obtained by E. D. Popova and I. Elishakoff on the basis of both the classical interval model and the new algebraic interval model, the latter recently proposed by E. D. Popova.

Keywords Intervals · Interval analysis · Interval arithmetic · Interval variables · Uncertain variables · Uncertainty · Systems of linear equations · Parametric interval systems · Equilibrium equations · Bars · Trusses · Loads · Reactions · Quantifiers · Universal quantifier · Existential quantifier · Quantified formulae · Quantified/free variables · Quantifier elimination · Quantifier-free formulae · Solution sets · Symbolic computations · Computer algebra systems · *Mathematica*

1. Introduction

1.1. Symbolic computations, computer algebra systems and quantifier elimination

Symbolic computations performed with the help of computer algebra systems (of course, together with arbitrary-precision numerical computations and graphical representations) have been successfully applied to a very large number of applied-mechanics problems since the middle 1960s. Such early computer algebra systems are *ALTRAN* (1965), *Reduce* (1966) and *Macsyma* (1968) whereas today the two most powerful such systems are *Maple* (1982) and *Mathematica* (1988). An interesting review concerning applied–structural mechanics was prepared by Pavlović [1] and published in 2003. In our recent research results (since 2016) including the present results, we mainly used *Mathematica* for the symbolic computations related to the method of quantifier elimination.

Quantifier elimination in elementary real algebra is a very interesting and algorithmically rather recent computational method in symbolic computations. The aim of this method is to eliminate the appearance of the well-known universal quantifier $\forall$ (for all) and existential quantifier $\exists$ (exists) in formulae including variables with at least one of these quantifiers (e.g. $\forall x$ or $\exists x$), called quantified variables, and to derive completely equivalent formulae from the mathematical–logical points of view, but free from these quantifiers and including only the free variables in the quantified formulae.

The most popular, efficient and general-purpose algorithm for quantifier elimination is CAD (cylindrical algebraic decomposition) although additional interesting quantifier elimination algorithms are also available and useful in some special cases. The CAD ingenious algorithm was devised by Collins in 1973 (at first, it was presented at a symposium held at Carnegie-Mellon University) with its first official and complete publication by Collins having appeared in 1975 [2]. The standard book on quantifier elimination and CAD is still the book edited by Caviness and Johnson [3] devoted to quantifier elimination and CAD and published (with some delay) in 1998. This book is based on a related symposium held at the Research Institute for Symbolic Computation in Linz, Austria (RISC-Linz) in October 1993 for the celebration of the 20th anniversary of CAD, but it includes all the fundamental research results going back to the fundamental original results on quantifier elimination by Tarski (during the period 1930–1951) at first officially published in 1948 and, next, in 1951. Additionally, a detailed bibliography on the applications of quantifier elimination in elementary real algebra was prepared by Ratschan [4] in 2012. On the other hand, a very large number of interesting research results on CAD by many authors is available in the literature; see, e.g., the recent papers on CAD by Strzeboński [5] and England, Bradford and Davenport [6].

Unfortunately, from the negative point of view it should be mentioned that as has been proved by Davenport and Heintz [7] and is now very well known, quantifier elimination for real variables has a doubly-exponential computational complexity [7]. Evidently, this serious disadvantage of the method of quantifier elimination constitutes a significant obstacle to its wide application especially to quantified formulae with a large total number of variables, i.e. both free and quantified variables.

Today, also unfortunately, it seems that only four general-purpose quantifier-elimination implementations are available and this happens in the following computer algebra systems: (i) *SACLIB* (with the famous `QEPCAD`, Quantifier Elimination by Partial Cylindrical Algebraic Decomposition, package mainly by Hong under the guidance of Collins, but also with contributions by several researchers), (ii) *Reduce* (with the `Redlog`, *Reduce* logic, package mainly by Dolzmann and Sturm), (iii) *Maple* (with the package `SyNRAC` mainly by Anai and Yanami) and (iv) *Mathematica* (mainly by Strzeboński by using the `Resolve` command devoted to quantifier elimination or, preferably, the `Reduce` general-purpose command of its kernel with the latter command exclusively used here).

With respect to the powerful and user-friendly *Mathematica* [8] implementation of quantifier elimination by Strzeboński, which will be exclusively used here, the main references are the chapter on real polynomial systems in the Wolfram monograph [9] on advanced topics in algebra as well as the related pages in the book by Trott [10, pp. 60–78], which is devoted to symbolic computations.

Following many researchers in various research fields, since 1994 the author has been interested in the application of the method of quantifier elimination to several problems of applied mechanics (see, e.g., Refs. [11, 12]). Much more recent applied-mechanics results by the author based on the same computational method, quantifier elimination, can be found in Refs. [13–29]. Moreover, the direct application of CAD in an applied-mechanics problem (optimal solutions to truss problems in structural mechanics) was recently successfully made by Charalampakis and Chatzigiannelis [30].

1.2. Interval analysis, its engineering applications and the overestimation phenomenon

Now, as far as interval analysis is concerned, its modern era began in 1959 with the initiation of publication of the related famous results by Moore and his collaborators; see, e.g., Refs. [31–33] although previous related results due to several authors beginning in the antiquity with Archimedes for the computation of the number π and reaching the much more recent results by Sunaga (in 1958) were already available. Additionally, a recent bibliography on interval computations and reliable computing including 784 entries was prepared by Beebe, Kearfott and Kreinovich [34] in 2017.

Commands concerning interval arithmetic are also available in *Mathematica* [8], e.g. the commands `Interval`, `IntervalUnion` and `IntervalIntersection`. These commands are described by Keiper [35]. Moreover, a useful *Mathematica* package, the package `directed.m`, concerning directed interval arithmetic and, hence, supporting its algebraic extension called Kaucher interval arithmetic, was prepared by Popova and Ullrich [36]. A second interesting *Mathematica* package, the package `IntervalComputations‘LinearSystems’`, devoted to the solution of systems of parametric and non-parametric interval linear algebraic equations, was also prepared by Popova [37].

It is also well known long ago that interval analysis has proved to be an extremely useful tool in applied mechanics during the last thirty years. Among an extremely large number of related interesting publications see, e.g., the papers (in chronological order) by Dimarogonas [38], Qiu, Chen and Song [39], Qiu and Elishakoff [40], Kulpa, Pownuk and Skalna [41], Guo and Lü [42], Skalna [43], Muhanna, Zhang and Mullen [44], Shao and Su [45], Elishakoff and Ohsaki [46] (book), Elishakoff and Miglis [47, 48], Verhaeghe, Desmet, Vandepitte and Moens [49], Wang and Qiu [50], Gabriele and Varano [51], Behera [52] (Ph.D. Thesis), Sofi, Muscolino and Elishakoff [53], Qiu and Wang [54], Zieniuk, Kapturczak and Kużelewski [55], Chakraverty, Hladík and Behera [56], Muscolino, Sofi and Giunta [57], Dinh-Cong, Van Hoa and Nguyen-Thoi [58], Faes and Moens [59, 60], Muscolino and Santoro [61], Sofi, Romeo, Barrera and Cocks [62], Sofi, Muscolino and Giunta [63], Skalna and Hladík [64] and, recently, Behera and Chakraverty [65].

A serious problem in the application of classical interval analysis consists in the overestimation phenomenon, i.e. in the computation of intervals wider (and, frequently, much wider) than the correct intervals in the case of dependence among the parameters of the problem under consideration. This inconvenient situation, which frequently leads to practically unacceptable intervals especially from the engineering point of view, is simply due to the fact that the rules of interval arithmetic assume the complete independence of the parameters of the problem although, quite frequently, this is not the case especially in applied-mechanics problems and engineering problems in general.

For the avoidance of this phenomenon following many related previous efforts, Elishakoff, Gabriele and Wang [66] recently suggested a modification to classical interval arithmetic as far as the subtraction of two interval quantities is concerned in the particular problem of computation of the resultant of several collinear interval forces. This modification of classical interval arithmetic was further elaborated and significantly and carefully extended by Popova [67–71], who also used generalized (both proper and improper) intervals appearing in the well-known Kaucher interval arithmetic. In this way, a completely new model for the avoidance of interval overestimations was developed by Popova [67–71] and it was called *algebraic interval model* (or *interval algebraic model*) because it is based on the algebraic (or formal) solutions to the related systems of linear equilibrium equations. This new and, naturally, interesting approach, including two simple but efficient numerical methods for the derivation of the algebraic (formal) solutions to the related systems of parametric interval linear equilibrium equations [71], was successfully applied to many applied-mechanics problems by Popova [67–71] and, quite recently, by Popova and Elishakoff [72, 73].

1.3. Relationship between interval analysis and quantifier elimination

It is very well known that quantifiers and quantifier elimination are strongly related to interval analysis. This seems to be natural if we take into account that several problems in interval analysis (such as the solution sets of parametric or non-parametric interval systems of linear algebraic equations) are expressed in terms of quantified formulae. Among many related results here we can make reference to the results by Grandón and Neveu [74], Grandón and Goldsztejn [75] and Khanh and Ogawa [76]. Moreover, Elishakoff, Gabriele and Wang [66] repeatedly used quantifiers during their study of the generalized Galilei's problem [66], e.g. at the end of Section 2 there [66, p. 1207] in order to provide a physical meaning to a simple interval equation with a generalized interval.

At this point we can also mention that the aforementioned implementation of quantifier elimination by Strzeboński in *Mathematica* [8] was successfully employed by Popova [77] as well as by Popova and Krämer [78] for the characterization of solution sets of parametric interval systems of linear algebraic equations. But, on the other hand, unfortunately, the derived results required too much CPU (central processing unit) time [77] in the computer used or they contained a very large number of logical expressions [78] in comparison with the same authors' efficient methods for the same computational tasks with respect to parametric interval systems of linear algebraic equations.

In eight recent technical reports [22–29], the author also combined quantifier elimination (continuously using its powerful implementation by Strzeboński in *Mathematica* [8]) with interval analysis in the following problems, almost all of which concern applied mechanics: (i) the computation of ranges of functions that appear in problems of applied mechanics [22], (ii) the determination of ranges of values of stress concentration factors in plane elasticity, more explicitly in notch and hole problems [23], (iii) similarly, for the ranges of values of stress intensity factors at crack tips in plane elasticity problems related to fracture mechanics [24], (iv) the derivation of sharp enclosures of the real roots of the classical parametric quadratic equation but with only one interval coefficient [25], (v) the determination of sharp bounds in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters [26], (vi) the derivation of symbolic intervals in simple problems of applied mechanics [27], (vii) the computation of intervals in three direct and inverse applied mechanics problems, more explicitly, a classical beam problem, a problem of a beam on a Winkler elastic foundation and the problem of free vibrations of the classical damped harmonic oscillator under critical damping [28] and, very recently, (viii) the determination of intervals for the resultants of interval forces satisfying existentially and/or universally quantified formulae [29].

As far as systems of parametric interval linear algebraic equations in interval analysis are concerned, these were studied in the related literature of interval analysis long ago; see, e.g., the recent interesting book by Skalna [79] on this class of systems, which also includes two applied mechanics applications. Among the solution sets of such systems the three best known are (i) the united solution set, which was the first set that was studied and it remains the most popular and useful of all, (ii) the tolerable solution set and (iii) the controllable solution set, but more general solution sets that depend on the quantifier used for each quantified variable, i.e. either the universal or the existential quantifier, have been considered. Additionally, among an extremely large number of publications related to generalized solution sets of interval equations (generally, but not always, systems of interval linear algebraic equations) here we can make reference to the interesting papers by Shary [80–84], Beaumont and Philippe [85], Goldsztejn [86], Goldsztejn and Chabert [87], Popova [88], Popova and Hladík [89] and also Dymova, Sevastjanov, Pownuk and Kreinovich [90].

1.4. Contents and aims of this technical report

Beyond the present introductory section, the present technical report is organized as follows:

- In Section 2, we consider the problem of a simply-supported truss with two external loads (Fig. 1). This problem was already proposed and solved by Popova [70, Section 4.1, pp. 225–227] using both the classical interval model and, mainly, her new algebraic interval model [67–71]. Here in this truss problem, by using the related system of parametric interval linear equilibrium equations and appropriate (existentially and/or universally) related quantified formulae with the present approach based on quantifier elimination (continuously in its efficient implementation in the computer algebra system *Mathematica* [8]), we derive (i) the related solution sets for the two unknown reactions at the two supports of the truss and a force on a concrete bar of the truss and (ii) the intervals for the same three unknown quantities (the reactions and the force). Several cases of use of the universal quantifier $\forall$ and/or the existential quantifier $\exists$ are considered. Moreover, the derived intervals for the three unknown quantities are compared with the corresponding intervals derived by Popova [70, Section 4.1, pp. 226–227] by the classical and the algebraic interval models.
- Next, in Section 3, we similarly study the problem of a clamped bar with a gap subjected to a concentrated load (Fig. 2) with the parameters of this mechanical problem represented again by intervals. The same problem was recently studied by Popova and Elishakoff [73, Section 3] by using again both the classical interval model and, mainly, Popova's new algebraic interval model [67–71].
- Finally, Section 4 constitutes a summary of the present results, where several conclusions are also drawn, a related discussion is made and a point requiring further investigation is mentioned.

The primary aim of this technical report is the extension of our previous results concerning the application of quantifier elimination to the derivation of sharp bounds in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters [26] from the purely existential case in the quantified formulae for systems of parametric interval linear equilibrium equations to the general case of quantified formulae including both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists). As is clear and will be seen below, such generalizations necessarily lead to narrower intervals of the unknown quantities (here mainly reactions) and, frequently, to degenerate intervals (deterministic values) or to no intervals at all, i.e. to the empty set $\emptyset$. It is well known and easily understood that the replacement of the existential quantifier by the universal quantifier in a quantified formula generally makes the satisfaction of this formula more difficult and, hence, the resulting intervals of the unknown quantities narrower and, frequently, degenerate or even non-existent (equal to the empty set). This situation is reasonable and expected.

Of course, because of the well-known doubly-exponential computational complexity of quantifier elimination proved by Davenport and Heintz [7], the present approach is applicable only to simple systems of parametric interval linear algebraic equations [79], i.e. to systems with a small number of total variables (both free and quantified variables) in the quantified formulae. Naturally, this situation constitutes a disadvantage of the present computational approach in its practical use.

On the other hand, here we compare the present intervals with those already obtained by Popova and Elishakoff (including, in few cases, degenerate intervals, i.e. crisp, deterministic values) and, frequently, we observe that under appropriate quantifications of the parameters of the mechanical problem in the quantified formulae the present intervals coincide with those already having been computed by Popova [70, Section 4.1, pp. 225–227] and by Popova and Elishakoff [73, Section 3].

But, unfortunately, we failed in the realization of our wish to determine under what conditions (i.e. here under what quantifications of the parameters of the mechanical system under consideration: universal and/or existential) such coincidences of intervals occur. Nevertheless, we still hope that this situation is not intractable and, therefore, it might be clarified in the near future. In this author's opinion, this will be very helpful to the engineer and will make Popova's recent algebraic interval model much more popular in applications of applied mechanics and engineering in general.

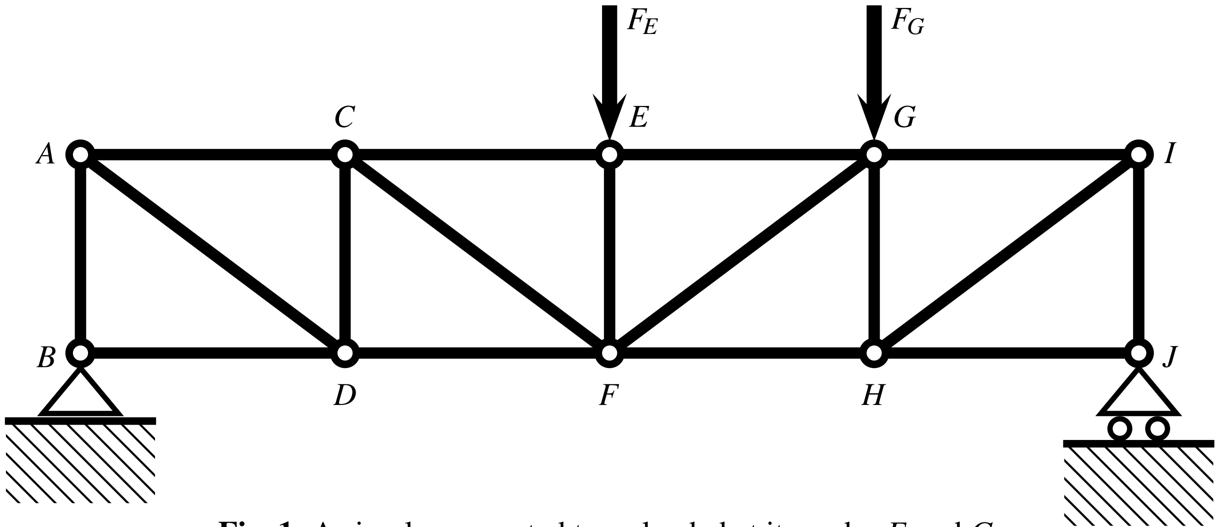

Fig. 1. A simply-supported truss loaded at its nodes E and G.

2. A simply-supported truss with two external loads

2.1. The truss problem and the related system of parametric interval linear equilibrium equations

In this section, we consider the problem of a simply-supported truss with seventeen bars under the action of two vertical external loads applied at its nodes E and G. This truss is shown in Fig. 1. The present truss problem was proposed and solved by Popova [70, Section 4.1, pp. 225–227]. The lengths of the eight horizontal bars of the present truss are equal to $L_h = 2.4$ [70, p. 225] (all the lengths are measured in meters, m) whereas the lengths of the five vertical bars of the truss are equal to $L_v = 1.8$ [70, p. 225]. Hence, the lengths of the four oblique bars of the truss are equal to

$$L_o = \sqrt{L_h^2 + L_v^2} = \sqrt{2.4^2 + 1.8^2} = 3 \tag{1}$$

(Fig. 1). The two vertical external loads F_E and F_G applied to the truss at its nodes E and G (Fig. 1), respectively, are assumed to be uncertain variables here interval-valued variables or, better, interval loads with [70, p. 225]

$$F_E \in [10.8, 13.2] \quad \text{and also} \quad F_G \in [10.8, 13.2] \tag{2}$$

with mean values equal to 12 [70, p. 225]. (All the forces are measured in kilonewtons, kN.)

By appropriately using elementary equilibrium equations in applied and structural mechanics with respect to the moments and forces, we directly obtain a system of three parametric interval linear equilibrium equations for the reactions R_B and R_J at the two supports B and J of the truss (Fig. 1), respectively, as well as for the force F_{FG} on the bar FG of the truss (Fig. 1), in which, following Popova [70, p. 226], we are also interested here. This system of equations has the form

$$F_E L_{FJ} + F_G L_{HJ} - R_B L_{BJ} = 0, \tag{3}$$

$$R_B - F_E - F_G + R_J = 0, \tag{4}$$

$$-\frac{3}{5} F_{FG} - F_G + R_J = 0. \tag{5}$$

We denote this system of linear equations by the symbol $\mathscr{E}_1$ and in *Mathematica* [8] by the related symbol `eqs1`. In Eq. (3) (related to the equilibrium of moments on the truss), the length symbols

$$L_{FJ} = 2L_h = 2 \times 2.4 = 4.8, \quad L_{HJ} = L_h = 2.4 \quad \text{and} \quad L_{BJ} = 4L_h = 4 \times 2.4 = 9.6 \tag{6}$$

denote the three distances between the two nodes F and J (for the distance L_{FJ}), H and J (for the distance L_{HJ}) and B and J (for the distance L_{BJ}) of the truss (Fig. 1), respectively. The same system of linear equations was also derived and used by Popova, but in its corresponding interval form in Kaucher interval arithmetic (with the dual operator) used by Popova [70, p. 226, Eqs. (5) and (6)].

It is very easy to solve the above system of linear equations (3)–(5) by using the well-known `Solve` command of *Mathematica* [8] for particular values of the two loads F_E and F_G (Fig. 1). By using as particular values the two endpoints 10.8 and 13.2 of the related intervals (2), i.e. the interval $[10.8, 13.2]$, we obtain the following values of the reactions R_B and R_J and the force F_{FG}:

$$R_B = 8.1, \quad R_J = 13.5, \quad F_{FG} = 4.5 \quad \text{for} \quad F_E = 10.8, \quad F_G = 10.8, \tag{7}$$

$$R_B = 8.7, \quad R_J = 15.3, \quad F_{FG} = 3.5 \quad \text{for} \quad F_E = 10.8, \quad F_G = 13.2, \tag{8}$$

$$R_B = 9.3, \quad R_J = 14.7, \quad F_{FG} = 6.5 \quad \text{for} \quad F_E = 13.2, \quad F_G = 10.8, \tag{9}$$

$$R_B = 9.9, \quad R_J = 16.5, \quad F_{FG} = 5.5 \quad \text{for} \quad F_E = 13.2, \quad F_G = 13.2. \tag{10}$$

Naturally, the above endpoint-related values of R_B, R_J and F_{FG} are very probable to appear below as endpoints of the intervals of the two reactions R_B and R_J on the present truss as well as of the force F_{FG} on the bar FG of the same truss (Fig. 1) and this is really the case as will be seen below using the method of quantifier elimination in its implementation by Strzeboński in *Mathematica* [8].

2.2. The solution sets of the system of parametric interval linear equilibrium equations

On the basis of the two equal intervals (2) of the external loads F_E and F_G on the truss of Fig. 1, we make the following assumptions $\mathscr{A}_1$ (denoted by the related symbol `ass1` in *Mathematica* [8]):

$$\mathscr{A}_1 = 10.8 \leq F_E \leq 13.2 \wedge 10.8 \leq F_G \leq 13.2. \tag{11}$$

Now we begin with the united solution set of the interval system $\mathscr{E}_1$ in Eqs. (3)–(5). For this popular and quite frequently used solution set we have the purely existentially quantified formula

$$\exists F_E \, \exists F_G \ \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1. \tag{12}$$

In order to perform quantifier elimination and to derive the related QFF (quantifier-free formula) we can use the quantifier elimination command (based on the `Reduce` command of *Mathematica*)

```
Reduce[Exists[{FE,FG}, ass1, eqs1], Reals]                                    [c1]
```

The resulting QFF has the approximate form (here we do not proceed to outward roundings)

$$\begin{aligned}
&(R_J = 13.5 \wedge R_B = 8.1 \wedge F_{FG} = 4.5) \\
&\quad \vee [13.5 < R_J \leq 14.7 \wedge 0.0666667(5R_J + 54) \leq R_B \leq 0.2(5R_J - 27) \\
&\qquad\qquad \wedge F_{FG} = 0.333333(10R_B - 5R_J)] \\
&\quad \vee [14.7 < R_J \leq 15.3 \wedge 0.0666667(5R_J + 54) \leq R_B \leq 0.0666667(5R_J + 66) \\
&\qquad\qquad \wedge F_{FG} = 0.333333(10R_B - 5R_J)] \\
&\quad \vee [15.3 < R_J < 16.5 \wedge 0.2(5R_J - 33) \leq R_B \leq 0.0666667(5R_J + 66) \\
&\qquad\qquad \wedge F_{FG} = 0.333333(10R_B - 5R_J)] \\
&\quad \vee (R_J = 16.5 \wedge R_B = 9.9 \wedge F_{FG} = 5.5).
\end{aligned} \tag{13}$$

In the above QFF, although both uncertain, interval loads F_E and F_G applied to the truss share the same interval $[10.8, 13.2]$, nevertheless, they were not assumed equal, i.e., in general, $F_E \neq F_G$. If we additionally assume that these two loads are equal, i.e. if we also assume that $F_E = F_G$, then we have the slightly modified but again purely existentially quantified formula

$$\begin{aligned}
&\exists F_E \, \exists F_G \ \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true} \\
&\qquad \text{under the assumptions } \mathscr{A}_1 \text{ and the additional assumption } F_E = F_G
\end{aligned} \tag{14}$$

and the related also slightly modified quantifier elimination command

```
Reduce[Exists[{FE,FG}, ass1 ∧ FE == FG, eqs1], Reals]//Simplify        [c2]
```

The resulting QFF (essentially again the related united solution set) has now the much simpler form

$$13.5 \le R_J \le 16.5 \wedge R_B = 0.6R_J \wedge F_{FG} = 1.66667R_J - 2.22222R_B\,, \tag{15}$$

but we directly observe that the interval of the reaction R_J, $R_J \in [13.5, 16.5]$, remains the same.

At this point we can add that by adopting the mixed universally–existentially quantified formula

$$\forall F_G\,\exists F_E \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1 \tag{16}$$

instead of the purely existentially similar quantified formula (12) previously (valid for the determination of the united solution set) and by using the related quantifier elimination command

```
Refine[Reduce[ForAll[FG, ass1, Exists[FE, ass1, eqs1]], Reals], ass1]        [c3]
```

we obtain the QFF `False`. This QFF corresponds to the empty set ∅ for the related new and now, additionally, generalized solution set. (In the above command [c3], the auxiliary `Refine` command of *Mathematica* [8] was also used so that the negation of the present assumptions $\mathscr{A}_1$ in Eq. (11) can be completely absent in the resulting QFF here the trivial QFF `False`.)

Finally, we can mention that by changing the order of the two quantified variables, i.e. of the two loads F_E and F_G in the above quantified formula (16), i.e. now with the load F_E universally quantified and the load F_G existentially quantified, we obtain again the same QFF, i.e. the QFF `False`, corresponding again to the empty set ∅ for the related new and again generalized solution set.

2.3. The separate intervals of the three unknown interval forces

Practically, as is well known in interval analysis, in interval systems of linear algebraic equations, we frequently prefer to derive just the intervals of the unknown quantities instead of the resulting solution sets, e.g. the popular united solution set. This will also be made in this subsection with respect to the three unknowns in the system $\mathscr{E}_1$ of linear equilibrium equations (3)–(5), i.e. the two reactions R_B and R_J at the supports B and J, respectively, and also the force F_{FG} on the bar FG of the truss of Fig. 1. Of course, for the derivation of the related intervals we will use again the method of quantifier elimination in its efficient implementation by Strzeboński in the popular computer algebra system *Mathematica* [8], which is exclusively used here.

2.3.1. The intervals of the reaction R_B at the left support B of the truss

We begin with the intervals of the reaction R_B at the left support B of the truss of Fig. 1. The two vertical loads F_E and F_G on the truss may be either universally or existentially quantified variables.

At first, in the purely existential case with quantified formula

$$\exists F_E\,\exists F_G\,\exists R_J\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{17}$$

we can use the related quantifier elimination command

```
Reduce[Exists[{FE,FG,RJ,FFG}, ass1,eqs1], Reals]        [c4]
```

Then we obtain the simple QFF (quantifier-free formula) for the sought interval of the reaction R_B

$$8.1 \le R_B \le 9.9, \quad \text{i.e.} \quad R_B \in [8.1, 9.9]. \tag{18}$$

This is exactly the interval of this reaction R_B already computed by Popova [70, p. 226] by using both (i) classical interval arithmetic and (ii) her new and surely interesting model based on her Theorem 1 [70, p. 222, Theorem 1] and, further, on the use of Kaucher arithmetic and generalized (both proper and improper) intervals. On the other hand, we observe that if we add the additional equality assumption $F_E = F_G$ for the two vertical external loads F_E and F_G applied to the truss to our assumptions $\mathscr{A}_1$ in Eq. (11), then we obtain the same interval $[8.1, 9.9]$ of the same reaction R_B.

Now we proceed to the subsequent cases where the universal quantifier $\forall$ (for all) is also present in the quantified formula either (i) for the load F_E at the node E of the truss or (ii) for the load F_G at the node G of the truss or, finally, (iii) for both these loads F_E and F_G applied to the truss of Fig. 1.

In these three cases, we have the three similar quantified formulae (with differences only in the quantifiers used for the loads F_E and F_G)

$$\forall F_E\,\exists F_G\,\exists R_J\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{19}$$

$$\forall F_G\,\exists F_E\,\exists R_J\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{20}$$

$$\forall F_E\,\forall F_G\,\exists R_J\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1. \tag{21}$$

The corresponding commands of *Mathematica* used here and the computed QFFs are

```
Refine[Reduce[ForAll[FE, ass1, Exists[{FG,RJ,FFG}, ass1, eqs1]],
        Reals], ass1]                                              [c5]
```

with resulting QFF simply `False`; next,

```
Refine[Reduce[ForAll[FG, ass1, Exists[{FE,RJ,FFG}, ass1, eqs1]],
        Reals], ass1]                                              [c6]
```

with resulting QFF

$$8.7 \le R_B \le 9.3, \quad \text{i.e.} \quad R_B \in [8.7, 9.3] \tag{22}$$

and, finally,

```
Refine[Reduce[ForAll[{FE,FG},ass1,Exists[{RJ,FFG},ass1,eqs1]],
        Reals],ass1]                                               [c7]
```

with resulting QFF `False` as is really expected in this purely universal (and, essentially, physically unacceptable) case with respect to the two interval loads F_E and F_G applied to the truss of Fig. 1.

On the other hand, by adding to our assumptions the equality constraint $F_E = F_G$ in the command [c6], we observe that now we get the QFF `False` instead of the QFF (22) previously (without this additional assumption), i.e. the related interval is the empty set $\emptyset$ instead of the interval (22).

Concluding, we observe that in the present truss problem of Fig. 1 and only with respect to the reaction R_B at the left support B of the truss, we obtain three different intervals, i.e. the intervals

$$R_{B1} = [8.1, 9.9], \quad R_{B2} = [8.7, 9.3] \quad \text{and} \quad R_{B3} = \emptyset \tag{23}$$

clearly with $R_{B2} \subset R_{B1}$. This is completely natural and expected since the first interval R_{B1} refers to the purely existentially quantified formula (17) (both loads F_E and F_G are existentially quantified) whereas the second interval R_{B2} (with $R_{B2} \subset R_{B1}$) refers to the mixed universally–existentially quantified formula (20) with the load F_G now universally quantified. Evidently, this makes the satisfaction of the formula more difficult and, therefore, the related interval of the reaction R_B narrower.

On the other hand, a similar and rather interesting remark is that the interval $R_{B2} = [8.7, 9.3]$ in the second of Eqs. (23) obtained in the QFF (22) is narrower than the interval $R_{B1} = [8.1, 9.9]$ in the first of Eqs. (23), which was originally obtained by Popova [70, p. 226] not only with classical interval arithmetic, but also with her new method also using Kaucher arithmetic. Unfortunately, we have been unable to explain this strange situation, i.e. the derivation of an ordinary interval, here the interval R_{B2}, narrower than the interval corresponding to the interval R_{B1} derived by using the algebraic interval model recently proposed by Popova [67–71]. It seems that the reply to this question will become clear when Popova's algebraic interval model is equivalently expressed through a related quantified formula if this might be possible, but this seems to be doubtful. It is hoped that the clarification of this important point in some sense is not very difficult and, hence, it will happen in the near future. But, on the other hand, it is understood that Popova's model based on Kaucher arithmetic does not correspond to an explicit solution set of the interval system of linear algebraic equations under consideration (frequently, equilibrium equations in applied mechanics as is here the case with the present system $\mathscr{E}_1$ in Eqs. (3)–(5) originally derived by Popova [70, p. 226]) contrary to what is the case, e.g., with the classical united, tolerable and controllable solution sets to interval systems of equations and various additional, generalized solution sets related to quantifiers.

2.3.2. The intervals of the reaction R_J at the right support J of the truss

In a completely analogous manner, we can now work for the second reaction R_J at the right support J of the truss of Fig. 1. Our assumptions remain the same: these are the assumptions $\mathscr{A}_1$ in Eq. (11), i.e. $10.8 \le F_E \le 13.2 \wedge 10.8 \le F_G \le 13.2$, exactly as the system of parametric interval linear equilibrium equations $\mathscr{E}_1$, i.e. Eqs. (3)–(5). The four quantified formulae for the intervals of this reaction R_J are similar to the formulae (17) and (19)–(21) previously and they have the forms

$$\exists F_E\,\exists F_G\,\exists R_B\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{24}$$

$$\forall F_E\,\exists F_G\,\exists R_B\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{25}$$

$$\forall F_G\,\exists F_E\,\exists R_B\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \tag{26}$$

$$\forall F_E\,\forall F_G\,\exists R_B\,\exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1. \tag{27}$$

By using the related quantifier elimination commands in *Mathematica*, we directly obtain the corresponding intervals of the reaction R_J at the support J as follows. At first, for the first quantified formula (24), a purely existentially quantified formula, we have the quantifier elimination command

```
Reduce[Exists[{FE,FG,RB,FFG}, ass1, eqs1], Reals]                    [c8]
```

with resulting QFF

$$13.5 \le R_J \le 16.5, \quad \text{i.e.} \quad R_J \in [13.5, 16.5]. \tag{28}$$

We observe that this is the interval also computed by Popova [70, p. 226] by using her algebraic interval approach, but, on the other hand, the interval $[11.7, 18.3]$ computed by Popova [70, p. 226] by using the classical interval approach has not been derived here. This simply seems to be an indication of the extremely well-known overestimation phenomenon when classical interval arithmetic is used, which is avoided by using Popova's algebraic interval approach [67–71], but also the present approach too whenever this is applicable as is here the case with the present truss problem (Fig. 1).

Next, for the second quantified formula (25) we have the quantifier elimination command

```
Refine[Reduce[ForAll[FE, ass1, Exists[{FG,RB,FFG}, ass1, eqs1]],
          Reals], ass1]                                                  [c9]
```

with resulting QFF

$$14.7 \le R_J \le 15.3, \quad \text{i.e.} \quad R_J \in [14.7, 15.3]. \tag{29}$$

Here we remark that this narrower interval of the reaction R_J, i.e. $[14.7, 15.3] \subset [13.5, 16.5]$, was not derived by Popova [70, p. 226] even by using her algebraic interval approach. Popova [70, p. 226] computed only the two intervals $[13.5, 16.5]$ and $[11.7, 18.3]$ by using the algebraic and the classical interval approaches, respectively, the first (and narrower) of which was also computed here in the QFF (28). (This interval is also related to the interval hull of the united solution set in Eq. (13).)

Finally, for the last two quantified formulae (26) and (27) we can use the two quantifier elimination commands in *Mathematica*

```
Refine[Reduce[ForAll[FG, ass1, Exists[{FE,RB,FFG}, ass1, eqs1]],
          Reals], ass1]                                                  [c10]
Refine[Reduce[ForAll[{FE,FG}, ass1, Exists[{RB,FFG}, ass1, eqs1]],
          Reals], ass1]                                                  [c11]
```

respectively, in both cases (with the latter being somewhat unrealistic) with resulting QFF `False`, i.e. the related interval is simply the empty set $\emptyset$. In other words, this result means that the two quantified formulae (26) and (27) cannot be satisfied by any solution of the parametric interval system of linear equilibrium equations $\mathscr{E}_1$ of the present truss problem, i.e. Eqs. (3)–(5).

Naturally, a final interesting case is again the case where the two external loads F_E and F_G at the nodes E and G, respectively, of the truss of Fig. 1 are equal, i.e. $F_E = F_G$. Under this additional assumption, in the purely existential case, we have the following quantified formula constituting a modification of the related quantified formula (24) now supplemented with this equality constraint:

$$\exists F_E\, \exists F_G\, \exists R_B\, \exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1 \text{ and the additional assumption } F_E = F_G. \quad (30)$$

The related quantifier elimination command in *Mathematica* has the rather simple form

```
Reduce[Exists[{FE,FG,RB,FFG}, ass1 ∧ FE == FG, eqs1], Reals]                [c12]
```

with resulting QFF again the QFF (28). Therefore, in the present purely existential case, the additional assumption $F_E = F_G$ (an equality constraint) has no influence at all on the resulting interval of the reaction R_J at the right support J of the truss of Fig. 1.

Under the same equality constraint $F_E = F_G$ completely different is the case with the mixed universally–existentially quantified formula (25). The related quantified formula has now the form

$$\forall F_E\, \exists F_G\, \exists R_B\, \exists F_{FG} \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1 \text{ and the additional assumption } F_E = F_G. \quad (31)$$

The corresponding quantifier elimination command in *Mathematica* now takes the form

```
Refine[Reduce[ForAll[FE, ass1, Exists[{FG,RB,FFG}, ass1 ∧ FE == FG, eqs1]],
        Reals], ass1]                                                       [c13]
```

with resulting QFF the trivial QFF `False` corresponding to the empty set $\emptyset$. This means that this additional assumption, $F_E = F_G$, has now a strong influence on the interval of the reaction R_J, which was originally the interval $[14.7, 15.3]$ in the QFF (29), but now it changed to the empty set $\emptyset$. This result makes the modified quantified formula (31) unsatisfiable contrary to the previous case with the quantified formula (30), where the additional assumption $F_E = F_G$ had no influence at all on the interval of the same reaction R_J.

2.3.3. The intervals of the force F_{FG} on the bar FG of the truss

Finally, in a completely analogous manner, following Popova [70, pp. 226–227], we also consider the force F_{FG} on the bar FG of the truss of Fig. 1 again under the assumptions $\mathscr{A}_1$ in Eq. (11), i.e. $10.8 \leq F_E \leq 13.2 \wedge 10.8 \leq F_G \leq 13.2$, exactly as the parametric interval system of linear equilibrium equations $\mathscr{E}_1$, i.e. Eqs. (3)–(5). Now our four quantified formulae for the intervals of this force F_{FG} are similar to the formulae (17) and (19)–(21) and also (24)–(27) and they have the forms

$$\exists F_E\, \exists F_G\, \exists R_B\, \exists R_J \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \quad (32)$$

$$\forall F_E\, \exists F_G\, \exists R_B\, \exists R_J \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \quad (33)$$

$$\forall F_G\, \exists F_E\, \exists R_B\, \exists R_J \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1, \quad (34)$$

$$\forall F_E\, \forall F_G\, \exists R_B\, \exists R_J \text{ such that the system of equations } \mathscr{E}_1 \text{ holds true under the assumptions } \mathscr{A}_1. \quad (35)$$

Now by using again the related quantifier elimination commands in *Mathematica*, we directly compute the corresponding intervals of this force F_{FG} (the force on the bar FG of the truss) as follows. At first, for the first quantified formula (32) we have the quantifier elimination command

```
Reduce[Exists[{FE,FG,RB,RJ}, ass1, eqs1], Reals]                            [c14]
```

with resulting QFF

$$3.5 \leq F_{FG} \leq 6.5, \quad \text{i.e.} \quad F_{FG} \in [3.5, 6.5]. \quad (36)$$

We observe that this is exactly the interval of this force F_{FG} also computed by Popova [70, p. 227] by appropriately using classical (standard) interval arithmetic as was expected.

On the other hand, by using the additional equality constraint $F_E = F_G$ (dependence of the two interval loads F_E and F_G) through the modified quantified formula

```
Reduce[Exists[{FE,FG,RB,RJ}, ass1 ∧ FE == FG, eqs1], Reals]        [c15]
```

we obtain the QFF

$$4.5 \le F_{FG} \le 5.5, \quad \text{i.e.} \quad F_{FG} \in [4.5, 5.5], \tag{37}$$

i.e. the narrower interval $[4.5, 5.5] \subset [3.5, 6.5]$ for the same force F_{FG}.

We remark that this is the interval also obtained by Popova [70, p. 226] by using her new algebraic interval approach [67–71] based on her Theorem 1 [70, p. 222] and, further, on generalized (both proper and improper) intervals and on Kaucher arithmetic, but without the additional assumption $F_E = F_G$ used here in the above command [c15]. Of course, the actual reason that the second interval $[4.5, 5.5]$ in the QFF (37) is narrower than the first interval $[3.5, 6.5]$ in the QFF (36) seems to be the additional assumption (here equality constraint) $F_E = F_G$ (dependence of these two external loads) and not the approach used for the computation of these two intervals. In fact, quantifier elimination completely avoids any overestimation (and underestimation too) of the resulting intervals contrary to what happens with classical interval arithmetic; see, e.g., the interval $[-2.5, 12.5]$ of this force F_{FG} [70, p. 226] computed for independent loads F_E and F_G compared to the aforementioned correct interval $[3.5, 6.5]$ for this case [70, p. 227] and also appearing in the QFF (36).

Next, for the second quantified formula (33) we have the quantifier elimination command

```
Refine[Reduce[ForAll[FE, ass1, Exists[{FG,RB,RJ}, ass1, eqs1]],
        Reals], ass1]                                              [c16]
```

with resulting QFF simply `False` corresponding to the empty set $\emptyset$ for the related interval.

Subsequently, for the third quantified formula (34) we have the quantifier elimination command

```
Refine[Reduce[ForAll[FG, ass1, Exists[{FE,RB,RJ}, ass1, eqs1]],
        Reals], ass1]                                              [c17]
```

with resulting QFF again the QFF (37) corresponding to the interval $[4.5, 5.5]$ of the same force F_{FG} on the bar FG of the truss of Fig. 1 also having been computed by Popova [70, p. 226] by using her algebraic interval approach (related to Kaucher generalized intervals) as was already mentioned.

At this point we can add that by additionally using the equality constraint $F_E = F_G$ (dependent external interval loads F_E and F_G) by using the modified quantifier elimination command

```
Refine[Reduce[ForAll[FG, ass1, Exists[{FE,RB,RJ}, ass1 ∧ FE == FG, eqs1]],
        Reals], ass1]                                              [c18]
```

we obtain the QFF `False` instead of the QFF (37) (essentially the interval $[4.5, 5.5]$) previously.

Finally, for the fourth quantified formula (35) we have the quantifier elimination command

```
Refine[Reduce[ForAll[{FE,FG}, ass1, Exists[{RB,RJ}, ass1, eqs1]],
        Reals], ass1]                                              [c19]
```

with resulting QFF simply `False` as is really expected in this purely universal and somewhat unrealistic case with respect to the two external interval loads F_E and F_G applied to the truss of Fig. 1.

On the other hand, we also used the following parametric formulae for the two loads F_E and F_G:

$$F_E(t) = 10.8 + (13.2 - 10.8)t, \quad F_G(t) = 10.8 + (13.2 - 10.8)t^2 \quad \text{with} \quad t \in [0, 1] \tag{38}$$

and with related interval the interval $[10.8, 13.2]$ for both these loads exactly as previously. Then we obtained again the interval $[4.5, 5.5]$ of the force F_{FG} on the bar FG of the truss of Fig. 1 exactly as in the QFF (37), which was derived with the simpler dependency $F_E = F_G$ between these two loads.

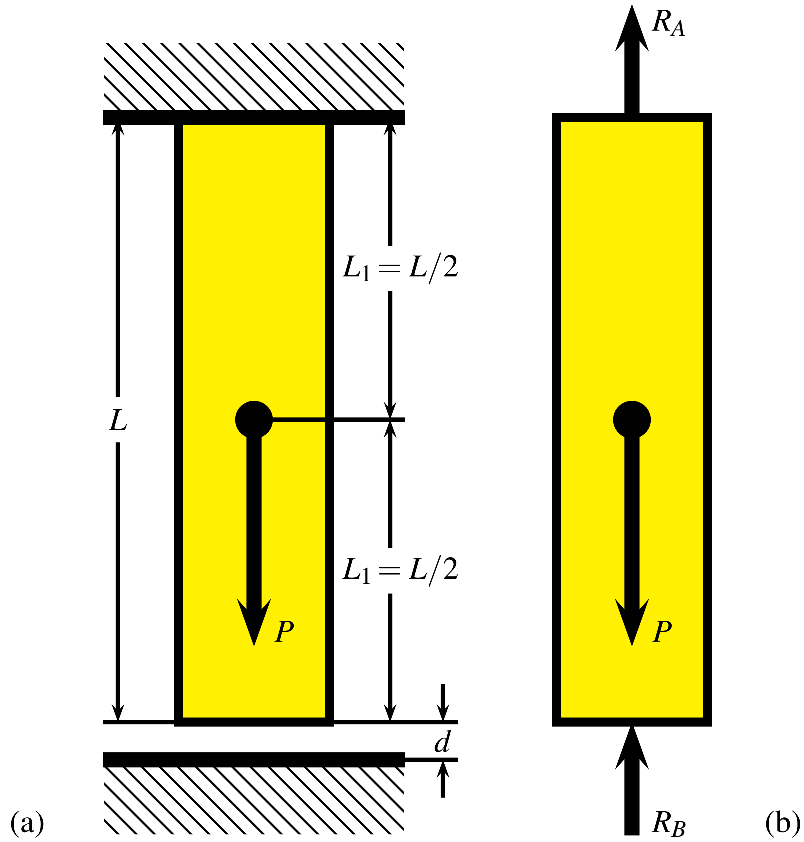

Fig. 2. (a) A clamped bar with a gap subjected to a concentrated load and (b) its free-body diagram.

3. A clamped bar with a gap subjected to a concentrated load

3.1. The bar problem and the related system of parametric interval linear equilibrium equations

In this section, we consider the problem of a bar clamped at its upper end and having a gap equal to d at its lower end subjected to a concentrated load P as is shown in Fig. 2. This problem was recently studied by Popova and Elishakoff [73, Section 3] again by using the algebraic interval approach recently proposed by Popova [67–71]. The modulus of elasticity (the Young modulus) of the isotropic elastic material of the bar is equal to E and the cross-sectional area of the bar is equal to A. The length of the present bar is equal to L and the concentrated load P is applied at the point L_1 of the bar (Fig. 2). Moreover, all the above quantities (i.e. the six quantities P, E, A, L, L_1 and d) are assumed to be uncertain variables with values varying within given intervals (interval-valued variables or simply interval variables) [73]. Moreover, following Popova and Elishakoff [73], for the reduction of the number of interval parameters and for simplicity in the computations we also assume that $L = 2L_1$ or, equivalently, $L_1 = L/2$ (Fig. 2). Additionally, Popova and Elishakoff [73, Eq. (12)] assumed the validity of the inequality constraint

$$d \leq \delta_A = \frac{PL_1}{EA} = \frac{PL}{2EA} = \frac{P}{k_A} \quad \text{and, equivalently,} \quad dk_A \leq P, \tag{39}$$

where δ_A denotes the elongation (the displacement) at the lower (the free) end of the bar and the stiffness k_A of the bar is defined by [73]

$$k_A := \frac{EA}{L_1} = \frac{2EA}{L}. \tag{40}$$

Then with the bar loaded by the concentrated load P (Fig. 2) there is no gap at the lower (free) end of the bar and the present interval problem is statically indeterminate [73]. Under these conditions we wish to find the two reactions R_A (at the upper end of the bar) and R_B (at the lower end of the bar), which, obviously, are also uncertain (here interval) variables.

This bar problem was already solved by Popova and Elishakoff [73, Section 3], who used both the classical and the algebraic interval approaches. The latter approach has been recently proposed by Popova [67–71] as has been already mentioned and, clearly, it leads to narrower intervals of both reactions R_A and R_B even to a degenerate interval (crisp value) of R_A [73, Section 3]. Additionally, Popova and Elishakoff [73, Section 3] computed the intervals of the axial stress σ on the bar using the algebraic interval model. Exactly as in the previous section for the truss problem of Fig. 1, here for the present bar problem of Fig. 2 we will determine the intervals of the reactions R_A and R_B by using the related existentially and/or universally quantified formulae and, next, the method of quantifier elimination continuously in the computational environment offered by *Mathematica* [8].

In the present simple bar problem (Fig. 2), the system of parametric linear algebraic equations has the following form [73, Eqs. (13)–(14)]:

$$R_A - P + R_B = 0, \quad \frac{PL_1}{EA} - \frac{R_B L}{EA} = d \tag{41}$$

with the first equation being the equilibrium equation and the second equation being the geometric compatibility equation [73, Eqs. (13)–(14)]. (We have also already assumed that $L_1 = L/2$; see also Fig. 2.) Now by using the stiffness k_A of the bar defined in Eq. (40), Eqs. (41) take their final form

$$R_A + R_B = P, \quad 2R_B = P - dk_A\,. \tag{42}$$

The closed-form solution of this simple system of parametric linear algebraic equations has the form

$$R_A = \frac{1}{2}\,(P + dk_A), \quad R_B = \frac{1}{2}\,(P - dk_A) \tag{43}$$

in agreement with the related interval-based results for this solution by Popova and Elishakoff [73].

Therefore, our constraints $\mathscr{C}$ in the present bar problem (Fig. 2) consist of the above two equations (equilibrium and geometric compatibility equations) (42) and the inequality constraint (39). Here these three constraints $\mathscr{C}$ are used in *Mathematica* in the following conjunctive logical form:

$$\mathscr{C} = R_A + R_B = P \wedge 2R_B = P - dk_A \wedge dk_A \le P \tag{44}$$

and they are denoted by the related symbol `cons` in *Mathematica*.

Now, following Popova and Elishakoff [73, Section 3, Example 1], we use the following mean values P_m, d_m, L_m, A_m and E_m of the five interval parameters P, d, L, A and E, respectively:

$$\begin{gathered} P_m = 200 \times 10^3 \text{ (in N)}, \quad d_m = 3 \times 10^{-4} \text{ (in m)}, \quad L_m = 3 \text{ (in m)}, \\ A_m = 25 \times 10^{-4} \text{ (in m}^2\text{)}, \quad E_m = 2 \times 10^{11} \text{ (in N/m}^2\text{)}. \end{gathered} \tag{45}$$

Additionally, following again Popova and Elishakoff [73, Section 3, Example 1], we assume that there is a 5% relative uncertainty in each of the above five interval parameters P, d, L, A and E. Therefore, it is convenient to additionally use the related auxiliary uncertain (here interval) variable

$$t \in [t_1, t_2] = \left[1 - \frac{5}{100}, 1 + \frac{5}{100}\right] = \left[1 - \frac{1}{20}, 1 + \frac{1}{20}\right] = \left[\frac{19}{20}, \frac{21}{20}\right] \quad \text{with} \quad t_1 = \frac{19}{20}, \quad t_2 = \frac{21}{20}. \tag{46}$$

Then we make the related obvious interval assumptions $\mathscr{A}_2$ (denoted by the related symbol `ass3` in *Mathematica*)

$$\begin{gathered} \mathscr{A}_2 = t_1 P_m \le P \le t_2 P_m \wedge t_1 d_m \le d \le t_2 d_m \wedge t_1 L_m \le L \le t_2 L_m \\ \wedge\, t_1 A_m \le A \le t_2 A_m \wedge t_1 E_m \le E \le t_2 E_m \end{gathered} \tag{47}$$

and using the numerical values of the mean values P_m, d_m, L_m, A_m and E_m in Eqs. (45) as well as of the endpoints t_1 and t_2 of the range $[t_1, t_2]$ of the auxiliary uncertain variable t in Eqs. (46), we get

$$\begin{gathered} \mathscr{A}_2 = 190000 \le P \le 210000 \wedge \frac{57}{200000} \le d \le \frac{63}{200000} \wedge \frac{57}{20} \le L \le \frac{63}{20} \\ \wedge\, \frac{19}{8000} \le A \le \frac{21}{8000} \ \wedge 19 \times 10^{10} \le E \le 21 \times 10^{10}. \end{gathered} \tag{48}$$

Now, by using the universally quantified formula

$$\forall\{P,d,L,A,E\} \text{ the inequality constraint (39) holds true under the assumptions } \mathscr{A}_2 \tag{49}$$

as well as the related quantifier elimination command in *Mathematica* (with E denoted as `E0`)

```
Reduce[ForAll[{P,d,L,A,E0}, ass2, cons[[3]]], Reals]    [c20]
```

we obtain the QFF (quantifier-free formula) `True`. This means that the inequality constraint (39) holds always true for the above interval values of the five uncertain parameters P, d, L, A and E in the present bar problem of Fig. 2. Nevertheless, here we will continue taking this constraint into account although, evidently, this is not necessary under the present assumptions $\mathscr{A}_2$ in Eq. (48).

For convenience, in our constraints $\mathscr{C}$ in Eq. (44), we have used the already defined (in Eq. (40)) stiffness $k_A := EA/L_1 = 2EA/L$ of the present bar as a single uncertain (here interval) parameter. This was also suggested by Popova and Elishakoff [73, Section 3]. For the range of this uncertain parameter k_A we can use the formula

$$k_{A_m} = \frac{2E_m A_m}{L_m} = \frac{10^9}{3} \approx 3.333333333 \times 10^8 \tag{50}$$

with respect to its mean value k_{A_m} and further the formula

$$\frac{t_1^2}{t_2} k_{A_m} \leq k_A \leq \frac{t_2^2}{t_1} k_{A_m} \tag{51}$$

with respect to the related range because of the definition $k_A := EA/L_1 = 2EA/L$ of the stiffness k_A of the bar in Eq. (40) and the uncertainty-related Eqs. (46). In this way, we directly find its range

$$\frac{1805 \times 10^7}{63} \leq k_A \leq \frac{735 \times 10^7}{19} \tag{52}$$

and approximately

$$2.865079365 \times 10^8 \leq k_A \leq 3.868421053 \times 10^8. \tag{53}$$

Exactly the same range can also be computed by using the related quantifier elimination command

```
Refine[Reduce[Exists[{E0,A,L}, ass2, kA == 2 E0 A/L], Reals], ass2]    [c21]
```

under the present assumptions $\mathscr{A}_2$ (`ass2` in *Mathematica*) appearing in their numerical form in Eq. (48).

Therefore, our assumptions $\mathscr{A}_2$ in Eq. (48) take now their modified, final form $\mathscr{A}_3$, which is

$$\mathscr{A}_3 = 190000 \leq P \leq 210000 \wedge \frac{57}{200000} \leq d \leq \frac{63}{200000} \wedge \frac{1805 \times 10^7}{63} \leq k_A \leq \frac{735 \times 10^7}{19} \tag{54}$$

and will be exclusively used below in this section. These assumptions $\mathscr{A}_3$ are denoted by the related symbol `ass3` in *Mathematica*.

3.2. The solution sets of the system of parametric interval linear equilibrium equations

We will use the above constraints $\mathscr{C}$ in Eq. (44) (here with only three parameters: P, d and k_A) including the equilibrium equation (first constraint), the geometric compatibility equation (second constraint) and the inequality constraint for the gap d at the lower end of the bar of Fig. 2 (third constraint) together with the above assumptions $\mathscr{A}_3$ in Eq. (54). At first, in the purely existential case, which concerns the so popular united solution set, the related quantified formula has the form

$$\exists P\, \exists d\, \exists k_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{55}$$

The related quantifier elimination command in *Mathematica* has the form

```
Refine[Reduce[Exists[{P,d,kA}, ass3, cons], {RA,RB}, Reals], ass3]//Expand [c22]
```

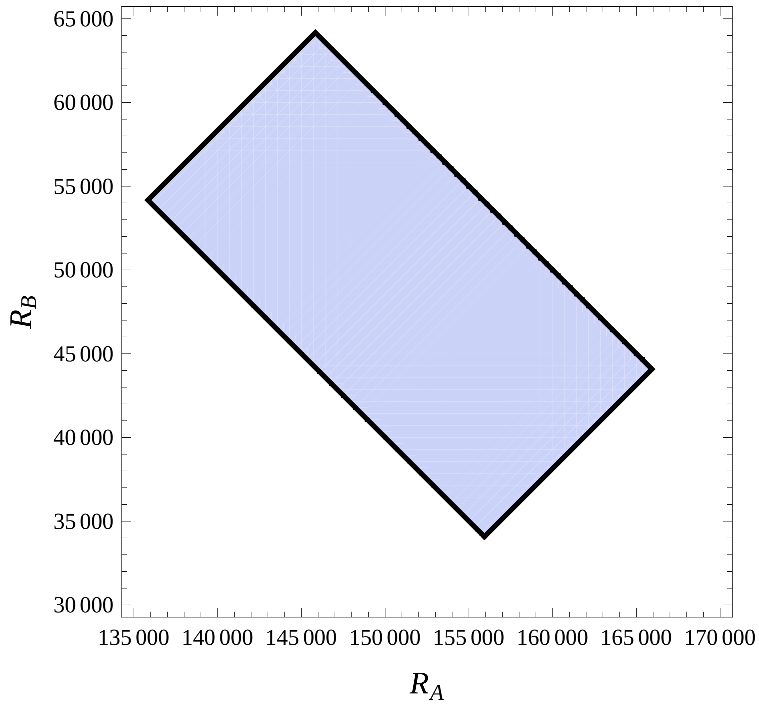

Fig. 3. The united solution set for the reactions R_A and R_B in the clamped-bar problem of Fig. 2.

The resulting QFF (quantifier-free formula), i.e. the united solution set for R_A and R_B, has the form

$$\begin{aligned}&\Big(R_A = \frac{2852375}{21} \wedge R_B = \frac{1137625}{21}\Big)\\ &\quad\vee \Big(\frac{2852375}{21} < R_A \le \frac{3062375}{21} \wedge 190000 - R_A \le R_B \le R_A - \frac{1714750}{21}\Big)\\ &\quad\vee \Big(\frac{3062375}{21} < R_A \le \frac{2962625}{19} \wedge 190000 - R_A \le R_B \le 210000 - R_A\Big)\\ &\quad\vee \Big(\frac{2962625}{19} < R_A < \frac{3152625}{19} \wedge R_A - \frac{2315250}{19} \le R_B \le 210000 - R_A\Big)\\ &\quad\vee \Big(R_A = \frac{3152625}{19} \wedge R_B = \frac{837375}{19}\Big)\end{aligned} \tag{56}$$

and numerically (approximately, but here with no outward roundings)

$$\begin{aligned}&(R_A = 135827.3810 \wedge R_B = 54172.61905)\\ &\quad\vee (135827.3810 < R_A \le 145827.3810 \wedge 190000 - R_A \le R_B \le R_A - 81654.76190)\\ &\quad\vee (145827.3810 < R_A \le 155927.6316 \wedge 190000 - R_A \le R_B \le 210000 - R_A)\\ &\quad\vee (155927.6316 < R_A < 165927.6316 \wedge R_A - 121855.2632 \le R_B \le 210000 - R_A)\\ &\quad\vee (R_A = 165927.6316 \wedge R_B = 44072.36842).\end{aligned} \tag{57}$$

The present solution set, i.e. here essentially the united solution set of the present parametric interval system of linear equilibrium equations, is displayed in the above Fig. 3. The same solution set is also directly seen to be in complete agreement with the corresponding set displayed by Popova and Elishakoff [73, Section 3] in the left Fig. 2 of this reference.

On the other hand, if the order of the two reactions R_A (at the upper, the clamped end of the bar) and R_B (at the lower end of the bar, the end with the gap d) is reversed during quantifier elimination, then by using the similar slightly modified quantifier elimination command

```
Refine[Reduce[Exists[{P,d,kA}, ass3, cons], {RB,RA}, Reals], ass3]//Expand [c23]
```

for the same united solution set, we get the completely equivalent QFF (quantifier-free formula)

$$\begin{aligned}&\Big(R_B = \frac{647375}{19} \wedge R_A = \frac{2962625}{19}\Big)\\ &\quad \vee \Big(\frac{647375}{19} < R_B \le \frac{837375}{19} \wedge 190000 - R_B \le R_A \le R_B + \frac{2315250}{19}\Big)\\ &\quad \vee \Big(\frac{837375}{19} < R_B \le \frac{1137625}{21} \wedge 190000 - R_B \le R_A \le 210000 - R_B\Big)\\ &\quad \vee \Big(\frac{1137625}{21} < R_B < \frac{1347625}{21} \wedge R_B + \frac{1714750}{21} \le R_A \le 210000 - R_B\Big)\\ &\quad \vee \Big(R_B = \frac{1347625}{21} \wedge R_A = \frac{3062375}{21}\Big) \qquad (58)\end{aligned}$$

and numerically (approximately, but again with no outward roundings)

$$\begin{aligned}&(R_B = 34072.36842 \wedge R_A = 155927.6316)\\ &\quad \vee (34072.36842 < R_B \le 44072.36842 \wedge 190000 - R_B \le R_A \le R_B + 121855.2632)\\ &\quad \vee (44072.36842 < R_B \le 54172.61905 \wedge 190000 - R_B \le R_A \le 210000 - R_B)\\ &\quad \vee (54172.61905 < R_B < 64172.61905 \wedge R_B + 81654.76190 \le R_A \le 210000 - R_B)\\ &\quad \vee (R_B = 64172.61905 \wedge R_A = 145827.3810). \qquad (59)\end{aligned}$$

Naturally, beyond the purely existential quantified formula (55) essentially corresponding to the united solution set as has been already mentioned, alternatively, we can use other, generalized quantified formulae now including both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists). For example, we can consider the following universally–existentially quantified formula:

$$\forall P \exists d \exists k_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \qquad (60)$$

The related quantifier elimination command in *Mathematica* now takes the form

```
Refine[Reduce[ForAll[P, ass3, Exists[{d,kA}, ass3, cons]], Reals], ass3] [c24]
```

now with resulting QFF (quantifier-free formula) simply `False` corresponding to the empty solution set $\emptyset$. This result simply means that the satisfaction of the quantified formula (60) is impossible under the present assumptions $\mathscr{A}_3$ already made in Eq. (54).

But, on the other hand, the similar quantified formula (now with the gap d at the lower end of the bar of Fig. 2 being the parameter that is universally quantified instead of the load P previously)

$$\forall d \exists P \exists k_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3 \qquad (61)$$

with related quantifier elimination command

```
Refine[Reduce[ForAll[d, ass3, Exists[{P,kA}, ass3, cons]], Reals], ass3] [c25]
```

leads to the non-trivial QFF (quantifier-free formula)

$$\begin{aligned}&(R_A = 140125 \wedge R_B = 49875)\\ &\quad \vee (140125 < R_A \le 150125 \wedge 190000 - R_A \le R_B \le R_A - 90250)\\ &\quad \vee (150125 < R_A < 160125 \wedge R_A - 110250 \le R_B \le 210000 - R_A)\\ &\quad \vee (R_A = 160125 \wedge R_B = 49875). \qquad (62)\end{aligned}$$

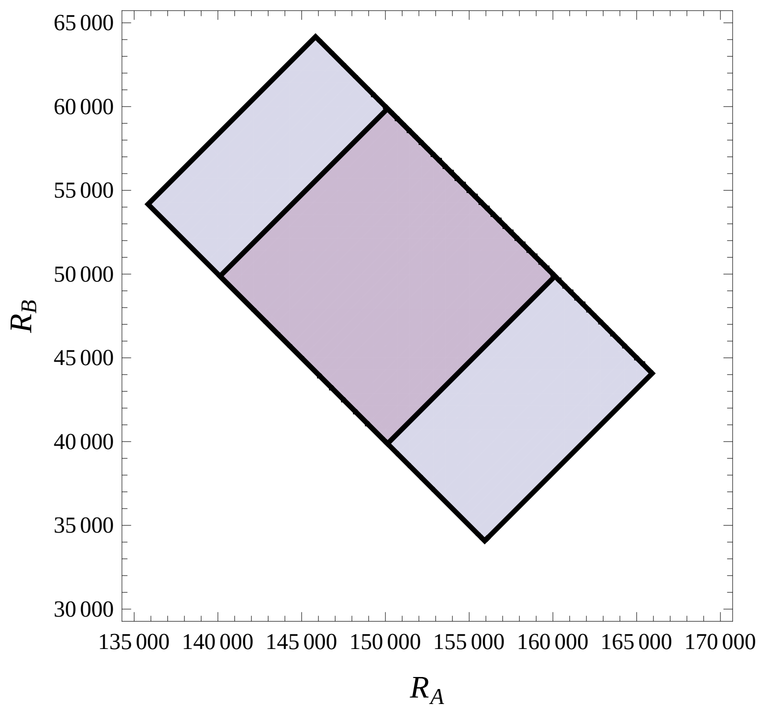

Fig. 4. The generalized solution set (internal solution set) in the QFF (62) for the two reactions R_A and R_B on the clamped bar of Fig. 2, which is a subset of the corresponding united solution set (external solution set) in the QFFs (56)–(59) also displayed in Fig. 3.

This QFF describes the present generalized solution set, which is displayed in the above Fig. 4 (internal solution set) together with the united solution set already displayed in Fig. 3. From Fig. 4 it is directly observed that the present generalized solution set in the above QFF (62) is a subset of the united solution set in the QFFs (56)–(59) already displayed in Fig. 3. This seems to be obvious because now the universal quantifier $\forall$ replaced the existential quantifier $\exists$ for the gap d at the lower end of the bar and this replacement made the satisfaction of the universally–existentially quantified formula (61) more difficult than the satisfaction of the purely existentially quantified formula (55).

Finally, the similar universally–existentially quantified formula (but now with the stiffness k_A of the bar universally quantified instead of the external load P or the gap d previously)

$$\forall k_A \, \exists P \, \exists d \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3 \tag{63}$$

with related quantifier elimination command

```
Refine[Reduce[ForAll[kA, ass3, Exists[{P,d}, ass3, cons]], Reals], ass3]   [c26]
```

leads again to the QFF `False`, i.e., essentially, to the empty solution set for the reactions R_A and R_B.

3.3. The separate intervals of the two unknown reactions R_A and R_B on the bar

Exactly as in the previous section, here we will further use the present quantifier-elimination-based approach in order to compute the intervals of the two unknown reactions R_A (at the upper, the clamped end of the bar of Fig. 2) and R_B (at the lower end of the same bar) separately, but, of course, based again on the constraints $\mathscr{C}$ in Eq. (44) for this bar and also on the assumptions $\mathscr{A}_3$ in Eq. (54). With respect to the united solution set displayed in Eqs. (56)–(59) as well as in Fig. 3 and in Fig. 4 (external solution set there) these intervals are seen to constitute the interval hull of this particular solution set, which encloses it. Of course, evidently, for the derivation of the present intervals of the two reactions R_A and R_B we will continue employing the method of quantifier elimination in its efficient implementation by Strzeboński in the popular computer algebra system *Mathematica* [8].

3.3.1. The intervals of the reaction R_A at the upper end (the clamped end) of the bar

We begin with the intervals of the reaction R_A at the upper end (the clamped end) of the bar of Fig. 2 under the constraints $\mathscr{C}$ in Eq. (44). We have again the three interval parameters P (external load), d (gap at the lower end of the bar) and k_A (stiffness of the bar). In the quantified formulae, these uncertain, interval parameters may be either universally or existentially quantified variables.

At first, we consider the purely existential case (with all three interval parameters P, d and k_A existentially quantified as is also the case with the second reaction R_B) with the quantified formula

$$\exists P \exists d \exists k_A \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{64}$$

The related quantifier elimination command has the form

```
Refine[Reduce[Exists[{P,d,kA,RB}, ass3, cons], Reals], ass3]                [c27]
```

with resulting QFF (here essentially the interval of the reaction R_A in this purely existential case)

$$\frac{2852375}{21} \le R_A \le \frac{3152625}{19} \quad \text{and numerically} \quad 135827.3810 \le R_A \le 165927.6316. \tag{65}$$

This interval is also the interval that was computed by Popova and Elishakoff [73, Example 1, first of Eqs. (17)] by using the classical interval model based on classical interval arithmetic. Since the present methodology based on quantifier elimination does never lead to the overestimation of an interval and, additionally, it is also based on the exact constraints (generally the equilibrium and compatibility equations) of the problem under consideration (here the problem of the bar of Fig. 2), the previous observation means that in the present bar problem the classical interval model does not lead to an overestimation of the interval of the reaction R_A as well contrary to what happens (the appearance of the overestimation phenomenon) in many other problems. This situation may be due to the possible use by Popova and Elishakoff of the closed-form formula $R_A = (1/2)(P + dk_A)$ [73, Example 1, first of Eqs. (17)] or of some equivalent formula, where each interval variable (here each parameter P, d and k_A) appears only once and, hence, the overestimation phenomenon is avoided.

Next, we consider the three cases with only one universally quantified parameter, but also two existentially quantified parameters. The related three separate quantified formulae have the forms

$$\forall P \exists d \exists k_A \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{66}$$

$$\forall d \exists P \exists k_A \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{67}$$

$$\forall k_A \exists P \exists d \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{68}$$

The corresponding quantifier elimination commands and the resulting QFFs (here again related to the intervals of the reaction R_A at the upper, the clamped end of the bar of Fig. 2) are

```
Refine[Reduce[ForAll[P, ass3, Exists[{d,kA,RB}, ass3, cons]], Reals], ass3] [c28]
```

and

$$\frac{3062375}{21} \le R_A \le \frac{2962625}{19} \quad \text{and numerically} \quad 145827.3810 \le R_A \le 155927.6316 \tag{69}$$

in the first case, i.e. that of the universally–existentially quantified formula (66). Next,

```
Refine[Reduce[ForAll[d, ass3, Exists[{P,kA,RB}, ass3, cons]], Reals], ass3] [c29]
```

and

$$140125 \le R_A \le 160125 \tag{70}$$

in the second case, i.e. that of the universally–existentially quantified formula (67). Finally,

```
Refine[Reduce[ForAll[kA, ass3, Exists[{P,d,RB}, ass3, cons]], Reals], ass3] [c30]
```

and

$$R_A = 150125 \tag{71}$$

in the third case, i.e. that of the universally–existentially quantified formula (68).

In this final case, we directly observe that contrary to the previous three cases, we obtain a crisp (a deterministic) value of the reaction R_A or, equivalently, a degenerate interval instead of an ordinary, a non-degenerate interval in the previous cases, i.e. in the three QFFs (65), (69) and (70).

Moreover, we observe that the above crisp value of the reaction R_A, $R_A = 150125$, in Eq. (71) is the value already computed by Popova and Elishakoff [73, Example 1] by using the new and undoubtedly interesting algebraic interval model having been recently proposed by Popova [67–71].

Remark: The coincidence of the crisp value $R_A = 150125$ in Eq. (71) computed by the present methodology with the crisp value originally computed by Popova and Elishakoff [73, Example 1] by using Popova's new algebraic interval model [67–71] by no means should be misinterpreted that these two approaches coincide. More explicitly, here the crisp value $R_A = 150125$ in Eq. (71) was based on the universally–existentially quantified formula (68) whereas the same value computed by Popova and Elishakoff was based on the algebraic (formal) solution of the related system of two interval linear equations [73, Eqs. (13) and (14)], i.e. it concerned a different mechanical problem from the mathematical–logical point of view. Unfortunately, so far this author has been unable to interpret Popova's new algebraic interval model [67–71] from the logical point of view (that related to the quantifiers $\forall$ and $\exists$) and, possibly, this is a non-trivial task or even it may be really impossible.

In a similar way, we can proceed to the more difficult case of two universally quantified parameters and only one existentially quantified parameter out of the three parameters P, d and k_A in the present bar problem of Fig. 2. The related three separate quantified formulae have now the forms

$$\forall P \forall d \exists k_A \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{72}$$

$$\forall P \forall k_A \exists d \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{73}$$

$$\forall d \forall k_A \exists P \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{74}$$

The corresponding quantifier elimination commands and the resulting QFFs (again related to the intervals of the reaction R_A if any) are

```
Refine[Reduce[ForAll[{P,d}, ass3, Exists[{kA,RB}, ass3, cons]], Reals], ass3] [c31]
```

and

$$R_A = 150125 \tag{75}$$

in the first case, i.e. that of the universally–existentially quantified formula (72).

In this case, we observe that we obtain a crisp (deterministic) value of the reaction R_A. We can also observe that this crisp value of the reaction R_A coincides with the value obtained in Eq. (71) for a different quantified formula there and it also coincides with the value computed by Popova and Elishakoff [73, Example 1] by using the new algebraic interval model proposed by Popova [67–71] as was already mentioned in the above remark. Next,

```
Refine[Reduce[ForAll[{P,kA}, ass3, Exists[{d,RB}, ass3, cons]], Reals], ass3] [c32]
```

with resulting QFF simply `False` in the second case, i.e. that of the universally–existentially quantified formula (73). Finally,

```
Refine[Reduce[ForAll[{d,kA}, ass3, Exists[{P,RB}, ass3, cons]], Reals], ass3] [c33]
```

again with resulting QFF simply `False` in the third case, i.e. that of the universally–existentially quantified formula (74).

The same QFF, `False`, is also obtained (as is really expected) with the quantified formula

$$\forall P \forall d \forall k_A \exists R_B \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3 \tag{76}$$

now with all three uncertain, interval parameters P, d and k_A in the present bar problem of Fig. 2 universally quantified and with related quantifier elimination command the following command:

```
Reduce[ForAll[{P,d,kA}, ass3, Exists[RB, ass3, cons]], Reals]          [c34]
```

Remark: From the above results for the intervals of the reaction R_A at the upper end of the bar of Fig. 2 it becomes completely clear that the replacement of the existential quantifier $\exists$ (exists) by the universal quantifier $\forall$ (for all) for a parameter (here P, d or k_A) in a related quantified formula generally makes the resulting interval narrower, never wider and, frequently, either degenerate (crisp value) or even equal to the empty set $\emptyset$. This is simply due to the fact that in a quantified formula the universal quantifier $\forall$ before a parameter requires the validity of this formula for all values of the parameter and, naturally, this is much more difficult to happen than the validity of the quantified formula for at least one value of the same parameter as is the case when the existential quantifier $\exists$ is used for the same parameter in the quantified formula.

3.3.2. The intervals of the reaction R_B at the lower end (the end with the gap) of the bar

Now we proceed to the computation of the intervals of the reaction R_B at the lower end of the bar again under the constraints $\mathscr{C}$ in Eq. (44). We still have the three interval parameters P (concentrated load), d (gap at the lower end of the bar) and k_A (stiffness of the bar). These interval parameters may again be either universally or existentially quantified variables in the quantified formulae.

At first, we consider again the purely existential case (the case with all three interval parameters P, d and k_A existentially quantified). This case corresponds to the quantified formula

$$\exists P \, \exists d \, \exists k_A \, \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{77}$$

The related quantifier elimination command has the following simple form (the use of the auxiliary `Refine` command permits again a simpler appearance of the resulting QFF):

```
Refine[Reduce[Exists[{P,d,kA,RA}, ass3, cons], Reals], ass3]          [c35]
```

with resulting QFF (here essentially the interval of the reaction R_B)

$$\frac{647375}{19} \le R_B \le \frac{1347625}{21} \quad \text{and numerically} \quad 34072.36842 \le R_B \le 64172.61905. \tag{78}$$

The above interval is again the interval which was already computed by Popova and Elishakoff [73, Example 1, second of Eqs. (17)] by using the classical interval model based on classical (standard) interval arithmetic without any overestimation exactly as happened previously too with the corresponding interval of the reaction R_A at the upper end of the bar of Fig. 2.

Next, we consider the three cases with only one universally quantified parameter, but also two existentially quantified parameters. The related three separate quantified formulae have the forms

$$\forall P \, \exists d \, \exists k_A \, \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{79}$$

$$\forall d \, \exists P \, \exists k_A \, \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{80}$$

$$\forall k_A \, \exists P \, \exists d \, \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{81}$$

The corresponding quantifier elimination commands in *Mathematica* and the resulting QFFs (again related to the intervals of the reaction R_B at the lower end of the bar of Fig. 2) are

```
Refine[Reduce[ForAll[P, ass3, Exists[{d,kA,RA}, ass3, cons]], Reals], ass3] [c36]
```

and

$$\frac{837375}{19} \le R_B \le \frac{1137625}{21} \quad \text{and numerically} \quad 44072.36842 \le R_B \le 54172.61905 \tag{82}$$

in the first case, i.e. that of the universally–existentially quantified formula (79). Next,

```
Refine[Reduce[ForAll[d, ass3, Exists[{P,kA,RA}, ass3, cons]], Reals], ass3] [c37]
```

and

$$39875 \le R_B \le 59875 \tag{83}$$

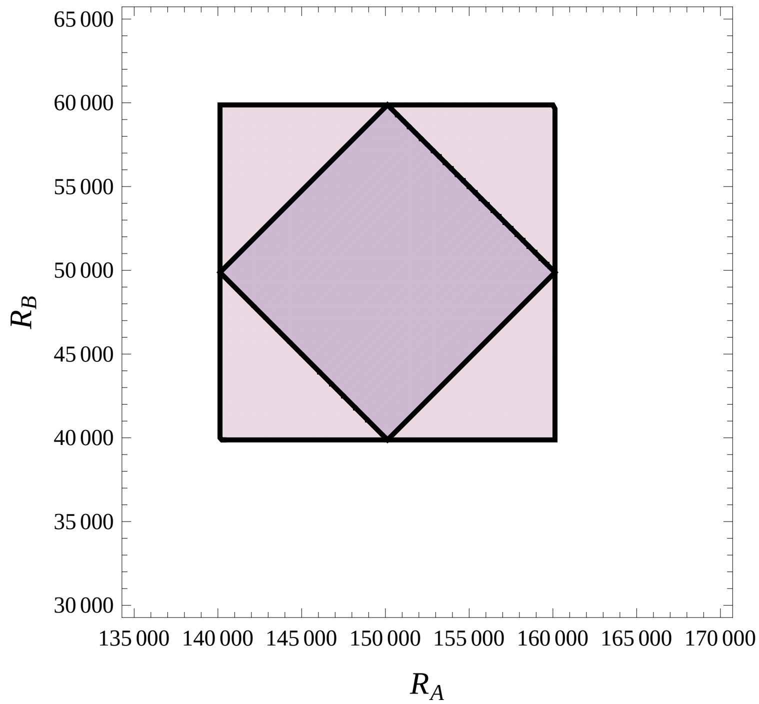

Fig. 5. The generalized solution set (internal solution set) in the QFF (62) (with the parameter d universally quantified and the parameters P and k_A existentially quantified) for the reactions R_A and R_B on the clamped bar of Fig. 2 and its interval hull based on the related QFFs (70) and (83).

in the second case, i.e. that of the universally–existentially quantified formula (80). We observe that this interval, $[39875, 59875]$, coincides with the interval of the reaction R_B computed by Popova and Elishakoff [73, Example 1] by using Popova's new algebraic interval model [67–71]. Finally,

```
Refine[Reduce[ForAll[kA, ass3, Exists[{P,d,RA}, ass3, cons]], Reals], ass3] [c38]
```

and

$$R_B = 49875 \tag{84}$$

in the third case, i.e. that of the universally–existentially quantified formula (81), where a crisp (a deterministic) value of the reaction R_B was computed.

In the above Fig. 5, we display the generalized solution set (internal solution set) in the QFF (62) (with the interval parameter d universally quantified and the interval parameters P and k_A existentially quantified) concerning the two reactions R_A and R_B on the clamped bar of Fig. 2. In the same figure (Fig. 5), we also display the interval hull of this generalized solution set. This interval hull is based on the QFF (70) for the reaction R_A and, simultaneously, on the QFF (83) for the reaction R_B.

Remark: We repeat that the coincidence of the interval $[39875, 59875]$ of the reaction R_B in the inequalities (83), which was computed by the present methodology based on quantifier elimination, with the related interval originally computed by Popova and Elishakoff [73, Example 1] based on Popova's new algebraic interval model [67–71] by no means should be misinterpreted that these two models coincide. In fact, they are completely different from the mathematical–logical point of view.

In a completely similar way, we can proceed to the case of two universally quantified parameters and only one existentially quantified parameter out of the three parameters P, d and k_A in the present bar problem of Fig. 2. The related three separate quantified formulae take now the forms

$$\forall P \forall d \exists k_A \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{85}$$

$$\forall P \forall k_A \exists d \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3, \tag{86}$$

$$\forall d \forall k_A \exists P \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3. \tag{87}$$

The corresponding quantifier elimination commands and the resulting QFFs (again related to the intervals of the reaction R_B if any at the lower end of the bar of Fig. 2) are

`Refine[Reduce[ForAll[{P,d}, ass3, Exists[{kA,RA}, ass3, cons]], Reals], ass3]` [c39]

in the first case, i.e. that of the universally–existentially quantified formula (85), with resulting QFF the previous QFF (84), which was already obtained with a different quantified formula.

Next, the following two quantifier elimination commands:

`Refine[Reduce[ForAll[{P,kA}, ass3, Exists[{d,RA}, ass3, cons]], Reals], ass3]` [c40]

in the second case, i.e. that of the universally–existentially quantified formula (86), and also

`Refine[Reduce[ForAll[{d,kA}, ass3, Exists[{P,RA}, ass3, cons]], Reals], ass3]` [c41]

in the third case, i.e. that of the universally–existentially quantified formula (87), yielded the same trivial QFF, the QFF `False`, corresponding to the empty set ∅.

The same trivial QFF is also obtained in the purely universal case, i.e. in the case with all three uncertain, interval parameters P, d and k_A universally quantified (only the reaction R_A, in which we are not interested here, remains existentially quantified), with the related quantified formula

$$\forall P \forall d \forall k_A \exists R_A \text{ such that the constraints } \mathscr{C} \text{ hold true under the assumptions } \mathscr{A}_3 \tag{88}$$

and the corresponding quantifier elimination command

`Reduce[ForAll[{P,d,kA}, ass3, Exists[RA, ass3, cons]], Reals]` [c42]

Clearly, the aforementioned resulting trivial QFF, `False`, was the expected QFF in the present case.

4. Conclusions – discussion

The modern computational method of quantifier elimination has been already successfully applied to several problems in mathematics, physics, engineering, control, biology, economics and other research areas. In the previous two sections, this method has been used for the determination of intervals related to the unknowns in systems of parametric interval linear algebraic equations resulting in applied mechanics problems and mainly concerning the related equilibrium equations.

Here the emphasis was put on (i) the determination of generalized solution sets for the unknown quantities (here mainly the reactions) in such systems with these solution sets computed with the help of the method of quantifier elimination, which was performed on quantified formulae including the universal quantifier ∀ (for all) and/or the existential quantifier ∃ (exists) and (ii) the determination of the intervals of each separate unknown quantity again on the basis of quantified formulae generally including the universal quantifier ∀ and/or the existential quantifier ∃, which is a practically more useful task and more easy to be performed from the computational point of view in comparison with the computation of a complete solution set. In the purely existential case for the well-known united solution set, it was observed that these separate intervals for the unknowns of the systems of parametric interval linear algebraic equations (here the equilibrium equations) determine the interval hull of this system of equations including its united solution set as a subset.

The two parametric applied mechanics problems studied here concerned (i) the problem of a simply-supported truss under the action of two external loads originally proposed and solved by Popova [70, Section 4.1, pp. 225–227] and here studied in Section 2 and (ii) the problem of a clamped bar (at its upper end) with a gap (at its lower end) subjected to a concentrated load originally proposed and solved by Popova and Elishakoff [73, Section 3] and here studied in Section 3.

For the solution of the above two parametric applied–structural mechanics problems, i.e. for the determination of the intervals of the unknown quantities (here forces and, more explicitly, mainly reactions), Popova [70, Section 4.1, pp. 225–227] and Popova and Elishakoff [73, Section 3] used both (i) the classical interval model based on classical (standard) interval arithmetic and (ii) the algebraic interval model having been recently proposed by Popova [67–71] and based on Kaucher interval arithmetic. Naturally, the latter model, Popova's algebraic interval model, generally leads to narrower intervals of the unknown uncertain (here interval) quantities in agreement with its aim to eliminate the very well-known, but very disturbing as well, overestimation phenomenon resulting in interval computations made on the basis of the rules for arithmetic in classical interval analysis.

In the previous two sections, it was observed that the intervals derived here by using the present approach based on quantifier elimination frequently coincide with the corresponding intervals already computed by Popova [70, Section 4.1, pp. 225–227] and Popova and Elishakoff [73, Section 3] in both cases of use of the classical or the algebraic interval models. Of course, this happens only when appropriate (for such a coincidence) quantified formulae are used, i.e. with appropriate quantifications of the parameters of the studied mechanical–structural system here a truss and a bar.

But on the other hand, unfortunately, the author failed to justify such coincidences of computed intervals of the unknown quantities between the present results and those by Popova and Elishakoff. This failure seems to be simply due to the fact that the present results are simply based on concrete quantified formulae (of course after the quantifier eliminations performed to these formulae) with the explicit use of the universal and/or the existential quantifiers in these formulae. On the contrary, the results derived by Popova and Elishakoff were based on the formal (algebraic) solutions of the related systems of parametric interval linear algebraic equations (here interval equilibrium equations), where no quantifiers are explicitly present. Clearly, this situation constitutes a weak point of the present results, probably a difficult or even a very difficult point as well, but it is hoped that in the future the interpretation of Popova's new algebraic interval model on the basis of quantified formulae may become possible. Alternatively, if this is impossible, then it is hoped that at least it will become possible to interpret Popova's algebraic interval model as providing useful intervals constituting inner/outer estimates of the corresponding intervals provided by the methods based on quantified formulae including the method of quantifier elimination exclusively used here. This seems to be a rather easy task with a series of related results being available in the literature of interval analysis long ago; see, e.g., the results by Shary [82] on the algebraic approach to systems of interval linear equations and many important additional results by Shary and some other authors.

The method of quantifier elimination applied here to general solution sets/intervals in parametric systems of interval linear equilibrium equations has the advantages that (i) it can be directly and interactively used in the computational environment offered by *Mathematica* [8] without any need to use special packages or difficult (at least for the engineer) programming, (ii) the obtained interval results for the unknown quantities have a clear mechanical–physical meaning from the mathematical–logical point of view without ambiguities because these results are based on concrete quantified formulae with the explicit use of the universal and/or existential quantifiers, (iii) the same results are free from any overestimation of the computed intervals (and underestimation too) and, finally, (iv) there is no need to use generalized intervals as is the case in Kaucher arithmetic.

But on the other hand, from the negative point of view, unfortunately, as is very well known, quantifier elimination for real variables has a doubly-exponential computational complexity [7]. Naturally, this disadvantage constitutes a significant obstacle to the wide application of quantifier elimination with a large total number of variables contrary to the present simple mechanical–structural applications, where small numbers of variables were present in the quantified formulae.

Finally, it should be emphasized that as is clear from the present results and, naturally, extremely well known and expected, the replacement of the existential quantifier $\exists$ by the universal quantifier $\forall$ for a quantified variable in a quantified formula causes (after the quantifier elimination) the derivation of a generally narrower interval for this variable. This situation is completely reasonable since it is more difficult to satisfy a formula valid for all values of a variable (when it is universally quantified) than for one (at least one) value of the same variable (when it is existentially quantified).

References

[1] Pavlović, M. N., Symbolic computation in structural engineering. *Computers & Structures*, **81** (22–23), 2121–2136 (2003). https://doi.org/10.1016/S0045-7949(03)00286-4

[2] Collins, G. E., Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In [3], pp. 85–121. https://doi.org/10.1007/978-3-7091-9459-1_4. Reproduced (with corrections) from the original publication (under the same title, Hauptvortrag) in: Brakhage, H. (editor), *Automata Theory and Formal Languages*, Proceedings of the 2nd GI Conference (Gesellschaft für Informatik), Kaiserslautern, Germany, May 20–23, 1975. Book series: *Lecture Notes in Computer Science (LNCS)*, Vol. 33, pp. 134–183. Springer, Berlin, 1975. https://doi.org/10.1007/3-540-07407-4_17

[3] Caviness, B. F. and Johnson, J. R. (editors), *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Springer, Wien, New York, 1998. https://doi.org/10.1007/978-3-7091-9459-1

[4] Ratschan, S., Applications of quantified constraint solving over the reals – bibliography. arXiv:1205.5571v1. Cornell University Library, Ithaca, New York, 2012. https://arxiv.org/abs/1205.5571v1

[5] Strzeboński, A., Cylindrical algebraic decomposition using local projections. *Journal of Symbolic Computation*, **76**, 36–64 (2016). https://doi.org/10.1016/j.jsc.2015.11.018

[6] England, M., Bradford, R. and Davenport, J. H., Cylindrical algebraic decomposition with equational constraints. *Journal of Symbolic Computation* (2020; in press; published online 26 July 2019). https://doi.org/10.1016/j.jsc.2019.07.019; also in arXiv:1903.08999. Cornell University Library, Ithaca, New York, 2019. https://arxiv.org/abs/1903.08999

[7] Davenport, J. H. and Heintz, J., Real quantifier elimination is doubly exponential. *Journal of Symbolic Computation*, **5** (1–2), 29–35 (1988). https://doi.org/10.1016/S0747-7171(88)80004-X

[8] Wolfram Research Inc., *Mathematica*, version 7.0.1. Wolfram Research Inc., Champaign, Illinois, 2009. https://www.wolfram.com

[9] Wolfram Research Inc., Wolfram Monograph: *Advanced Topics in Algebra: Real Polynomial Systems*. Wolfram Research Inc., Champaign, Illinois, 2014. https://reference.wolfram.com/language/tutorial/RealPolynomialSystems.html

[10] Trott, M., *The Mathematica GuideBook for Symbolics*. Springer, New York, 2006. https://doi.org/10.1007/0-387-28815-5

[11] Ioakimidis, N. I., Application of quantifier elimination to a simple elastic beam finite element below a straight rigid obstacle. *Mechanics Research Communications*, **22** (3), 271–278 (1995). https://doi.org/10.1016/0093-6413(95)00023-K

[12] Ioakimidis, N. I., Fracture initiation at an elastic crack tip: a computational implementation of the T-criterion. *International Journal of Fracture*, **98** (3–4), 293–311 (1999). https://doi.org/10.1023/A:1018662729282

[13] Ioakimidis, N. I., Derivation of conditions of complete contact for a beam on a tensionless Winkler elastic foundation with *Mathematica*. *Mechanics Research Communications*, **72**, 64–73 (2016). https://doi.org/10.1016/j.mechrescom.2016.01.007

[14] Ioakimidis, N. I., Caustics, pseudocaustics and the related illuminated and dark regions with the computational method of quantifier elimination. *Optics and Lasers in Engineering*, **88**, 280–300 (2017). https://doi.org/10.1016/j.optlaseng.2016.07.001

[15] Ioakimidis, N. I., Application of quantifier elimination to inverse buckling problems. *Acta Mechanica*, **228** (10), 3709–3724 (2017). https://doi.org/10.1007/s00707-017-1905-5

[16] Ioakimidis, N. I., Application of quantifier elimination to mixed-mode fracture criteria in crack problems. *Archive of Applied Mechanics*, **87** (10), 1567–1604 (2017). https://doi.org/10.1007/s00419-017-1274-5

[17] Ioakimidis, N. I., Supplementing the numerical solution of singular/hypersingular integral equations/inequalities with parametric inequality constraints with applications to crack problems. *Computer Assisted Methods in Engineering and Science*, **24** (1), 41–65 (2017). https://cames.ippt.gov.pl/index.php/cames/article/view/202

[18] Ioakimidis, N. I., The energy method in problems of buckling of bars with quantifier elimination. *Structures*, **13**, 47–65 (2018). https://doi.org/10.1016/j.istruc.2017.08.002

[19] Ioakimidis, N. I., Redlog-based derivation of feasibility conditions for singular integral equations of crack problems under parametric inequality constraints. Technical Report No. TR-2018-O5, School of Engineering, University of Patras, Patras, Greece, 2018, 14 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11217; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11217/1/TR-2018-O5.pdf; see also https://doi.org/10.13140/RG.2.2.15677.72164

[20] Ioakimidis, N. I., Application of quantifier elimination to inverse free vibration problems for inhomogeneous beams and bars. Technical Report No. TR-2018-Q1, School of Engineering, University of Patras, Patras, Greece, 2018, 22 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11410; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11410/1/TR-2018-Q1.pdf; see also https://doi.org/10.13140/RG.2.2.35810.38080

[21] Ioakimidis, N. I., Derivation of feasibility conditions in classical vibration problems by using the method of quantifier elimination. Technical Report No. TR-2018-Q2, School of Engineering, University of Patras, Patras, Greece, 2018, 16 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11505; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11505/1/TR-2018-Q2.pdf; see also https://doi.org/10.13140/RG.2.2.31616.07688

[22] Ioakimidis, N. I., Computation of ranges of functions in problems of applied mechanics with the computational method of quantifier elimination. Technical Report No. TR-2018-Q3, School of Engineering, University of Patras, Patras, Greece, 2018, 24 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11605; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11605/1/TR-2018-Q3.pdf; see also https://doi.org/10.13140/RG.2.2.13161.13929

[23] Ioakimidis, N. I., Stress concentration factors: determination of ranges of values by the method of quantifier elimination. Technical Report No. TR-2018-Q4, School of Engineering, University of Patras, Patras, Greece, 2018, 22 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11736; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11736/1/TR-2018-Q4.pdf; see also https://doi.org/10.13140/RG.2.2.19872.02566

[24] Ioakimidis, N. I., Interval computations in the formulae for the stress intensity factors at crack tips using the method of quantifier elimination. Technical Report No. TR-2019-Q5, School of Engineering, University of Patras, Patras, Greece, 2019, 20 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12153; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12153/1/TR-2019-Q5.pdf; see also https://doi.org/10.13140/RG.2.2.33293.79848

[25] Ioakimidis, N. I., Sharp enclosures of the real roots of the classical parametric quadratic equation with one interval coefficient by the method of quantifier elimination. Technical Report No. TR-2019-Q6, School of Engineering, University of Patras, Patras, Greece, 2019, 30 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12159; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12159/1/TR-2019-Q6.pdf; see also https://doi.org/10.13140/RG.2.2.16516.58248

[26] Ioakimidis, N. I., Sharp bounds based on quantifier elimination in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters. Technical Report No. TR-2019-Q7, School of Engineering, University of Patras, Patras, Greece, 2019, 45 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12497; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12497/1/TR-2019-Q7.pdf; see also https://doi.org/10.13140/RG.2.2.22662.52803

[27] Ioakimidis, N. I., Symbolic intervals for the unknown quantities in simple applied mechanics problems with the computational method of quantifier elimination. Technical Report No. TR-2019-Q8, School of Engineering, University of Patras, Patras, Greece, 2019, 26 pages. Web page at *Nemertes*, the

institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12745; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12745/1/TR-2019-Q8.pdf; see also https://doi.org/10.13140/RG.2.2.26311.24485

[28] Ioakimidis, N. I., Interval computations in various direct and inverse applied mechanics problems related to quantifiers by using the method of quantifier elimination. Technical Report No. TR-2019-Q9, School of Engineering, University of Patras, Patras, Greece, 2019, 34 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/13204; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/13204/1/TR-2019-Q9.pdf; see also https://doi.org/10.13140/RG.2.2.13976.14087

[29] Ioakimidis, N. I., Intervals for the resultants of interval forces with existentially and/or universally quantified formulae with the help of the method of quantifier elimination. Technical Report No. TR-2019-Q10, School of Engineering, University of Patras, Patras, Greece, 2019, 34 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras gr/jspui/handle/10889/13205; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/13205/12745/1/TR-2019-Q10.pdf; see also https://doi.org/10.13140/RG.2.2.11338.93125

[30] Charalampakis, A. E. and Chatzigiannelis, I., Analytical solutions for the minimum weight design of trusses by cylindrical algebraic decomposition. *Archive of Applied Mechanics*, **88** (1–2), 39–49 (2018). https://doi.org/10.1007/s00419-017-1271-8

[31] Moore, R. E., Automatic error analysis in digital computation. Technical Report LMSD-48421, Lockheed Aircraft Corporation, Missiles and Space Division, Sunnyvale, California, 1959, v+59 pages. https://interval.louisiana.edu/Moores_early_papers/Moore_Lockheed.pdf

[32] Moore, R. E., *Interval Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

[33] Moore, R. E., Kearfott, R. B. and Cloud, M. J., *Introduction to Interval Analysis*. SIAM (Society for Industrial and Applied Mathematics), Philadelphia, Pennsylvania, 2009. https://doi.org/10.1137/1.9780898717716

[34] Beebe, N. H. F., Kearfott, R. B. and Kreinovich, V., A bibliography of publications in *interval computations* and *reliable computing*, version 0.03, 2017. https://www.netlib.org/tex/bib/reliablecomputing.pdf

[35] Keiper, J. B., Interval arithmetic in Mathematica. *Interval Computations*, **3**, 76–87 (1993). https://interval.louisiana.edu/reliable-computing-journal/1993/interval-computations-1993-3-pp-076-087.pdf

[36] Popova, E. D. and Ullrich, Ch. P., Directed interval arithmetic in *Mathematica*: implementation and applications. Technical Report 96-3, Universitätsrechenzentrum und Institut für Informatik (URZ+IFI), University of Basel, Basel, Switzerland, 1996, i+43 pages. Web page of the related PDF file at the Bulgarian Academy of Sciences, Sofia, Bulgaria: http://www.math.bas.bg/˜epopova/papers/tr96-3.pdf

[37] Popova, E. D., Parametric interval linear solver. *Numerical Algorithms*, **37** (1–4), 345–356 (2004). https://doi.org/10.1023/B:NUMA.0000049480.57066.fa

[38] Dimarogonas, A. D., Interval analysis of vibrating systems. *Journal of Sound and Vibration*, **183** (4), 739–749 (1995). https://doi.org/10.1006/jsvi.1995.0283

[39] Qiu, Z., Chen, S. and Song, D., The displacement bound estimation for structures with an interval description of uncertain parameters. *Communications in Numerical Methods in Engineering*, **12** (1), 1–11 (1996). https://doi.org/10.1002/(SICI)1099-0887(199601)12:1<1::AID-CNM884>3.0.CO;2-N

[40] Qiu, Z. P. and Elishakoff, I., Antioptimization of structures with large uncertain-but-non-random parameters via interval analysis. *Computer Methods in Applied Mechanics and Engineering*, **152** (3–4), 361–372 (1998). https://doi.org/10.1016/S0045-7825(96)01211-X

[41] Kulpa, Z., Pownuk, A. and Skalna, I., Analysis of linear mechanical structures with uncertainties by means of interval methods. *Computer Assisted Mechanics and Engineering Sciences*, **5** (4), 443–477 (1998). https://www.infona.pl/resource/bwmeta1.element.baztech-article-BPB2-0001-0113; web page of the related PDF file at Semantic Scholar: https://pdfs.semanticscholar.org/f82c/6cb1de4f0c24bb63b1203ad85ca28993f 68e.pdf

[42] Guo, S.-x. and Lü, Z.-z., Interval arithmetic and static interval finite element method. *Applied Mathematics and Mechanics (English Edition)*, **22** (12), 1390–1396 (2001). https://doi.org/10.1007/BF02435542

[43] Skalna, I., Methods for solving systems of linear equations of structure mechanics with interval parameters. *Computer Assisted Mechanics and Engineering Sciences*, **10** (3), 281–293 (2003). https://www.infona.pl/resource/bwmeta1.element.baztech-article-BPB1-0009-0078; web page at *ResearchGate* with a link to the related PDF file: https://www.researchgate.net/publication/248687657

[44] Muhanna, R. L., Zhang, H. and Mullen, R. L., Interval finite elements as a basis for generalized models of uncertainty in engineering mechanics. *Reliable Computing*, **13**, 173–194 (2007). https://doi.org/10.1007/s11155-006-9024-3

[45] Shao, G.-j. and Su, J.-b., Interval finite element method and its application on anti-slide stability analysis. *Applied Mathematics and Mechanics (English Edition)*, **28** (4), 521–529 (2007). https://doi.org/10.1007/s10483-007-0413-y

[46] Elishakoff, I. and Ohsaki, M., *Optimization and Anti-Optimization of Structures Under Uncertainty*. Imperial College Press, London, 2010. https://doi.org/10.1142/9781848164789.

[47] Elishakoff, I. and Miglis, Y., Novel parameterized intervals may lead to sharp bounds. *Mechanics Research Communications*, **44**, 1–8 (2012). https://doi.org/10.1016/j.mechrescom.2012.04.004

[48] Elishakoff, I. and Miglis, Y., Overestimation-free computational version of interval analysis. *International Journal for Computational Methods in Engineering Science and Mechanics*, **13** (5), 319–328 (2012). https://doi.org/10.1080/15502287.2012.683134

[49] Verhaeghe, W., Desmet, W., Vandepitte, D. and Moens, D., Interval fields to represent uncertainty on the output side of a static FE analysis. *Computer Methods in Applied Mechanics and Engineering*, **260**, 50–62 (2013). https://doi.org/10.1016/j.cma.2013.03.021

[50] Wang, C. and Qiu, Z.-p., Equivalent method for accurate solution to linear interval equations. *Applied Mathematics and Mechanics (English Edition)*, **34** (8), 1031–1042 (2013). https://doi.org/10.1007/s10483-013-1725-6

[51] Gabriele, S. and Varano, V., Influence of the parameterization in the interval solution of elastic beams. *Journal of Structures*, Vol. 2014, Article ID 395213, 5 pages. https://doi.org/10.1155/2014/395213

[52] Behera, D., *Numerical Solution of Static and Dynamic Problems of Imprecisely Defined Structural Systems*. Ph.D. Thesis, Department of Mathematics, National Institute of Technology Rourkela, Rourkela, Odisha, India, 2014, xix+259 pages. http://ethesis.nitrkl.ac.in/6583; web page of the related PDF file: http://ethesis.nitrkl.ac.in/6583/1/Ph._D._Thesis_%40Diptiranjan_11-12-2014.pdf

[53] Sofi, A., Muscolino, G. and Elishakoff, I., Static response bounds of Timoshenko beams with spatially varying interval uncertainties. *Acta Mechanica*, **226** (11), 3737–3748 (2015). https://doi.org/10.1007/s00707-015-1400-9

[54] Qiu, Z.-P. and Wang, L., The need for introduction of non-probabilistic interval conceptions into structural analysis and design. *Science China – Physics, Mechanics & Astronomy*, **59** (11), Paper No. 114632 (2016). https://doi.org/10.1007/s11433-016-0329-3

[55] Zieniuk, E., Kapturczak, M. and Kużelewski, A., Solving interval systems of equations obtained during the numerical solution of boundary value problems. *Computational and Applied Mathematics*, **35** (2), 629–638 (2016). https://doi.org/10.1007/s40314-014-0209-9

[56] Chakraverty, S., Hladík, M. and Behera, D., Formal solution of an interval system of linear equations with an application in static responses of structures with interval forces. *Applied Mathematical Modelling*, **50**, 105–117 (2017). https://doi.org/10.1016/j.apm.2017.05.010

[57] Muscolino, G., Sofi, A. and Giunta, F., Dynamics of structures with uncertain-but-bounded parameters via pseudo-static sensitivity analysis. *Mechanical Systems and Signal Processing*, **111**, 1–22 (2018). https://doi.org/10.1016/j.ymssp.2018.02.023

[58] Dinh-Cong, D., Van Hoa, N. and Nguyen-Thoi, T., An effective optimization-based parameterized interval analysis approach for static structural response with multiple uncertain parameters. *Engineering with Computers*, 14 pages, **36** (2020; in press; published online 22 June 2019). https://doi.org/10.1007/s00366-019-00803-3

[59] Faes, M. and Moens, D., Multivariate dependent interval finite element analysis via convex hull pair constructions and the Extended Transformation Method. *Computer Methods in Applied Mechanics and Engineering*, **347**, 85–102 (2019). https://doi.org/10.1016/j.cma.2018.12.021

[60] Faes, M. and Moens, D., Recent trends in the modeling and quantification of non-probabilistic uncertainty. *Archives of Computational Methods in Engineering*, 39 pages, **27** (2020; in press; published online 23 February 2019). https://doi. org/10.1007/s11831-019-09327-x

[61] Muscolino, G. and Santoro, R., Dynamics of multiple cracked prismatic beams with uncertain-but-bounded depths under deterministic and stochastic loads. *Journal of Sound and Vibration*, **443**, 717–731 (2019). https://doi.org/10.1016/j.jsv.2018.11.029

[62] Sofi, A., Romeo, E., Barrera, O. and Cocks, A., An interval finite element method for the analysis of structures with spatially varying uncertainties. *Advances in Engineering Software*, **128**, 1–19 (2019). https://doi.org/10.1016/j.advengsoft.2018.11.001

[63] Sofi, A., Muscolino, G. and Giunta, F., Fatigue analysis of structures with interval axial stiffness subjected to stationary stochastic excitations. *Meccanica*, **54** (9), 1471–1487 (2019). https://doi.org/10.1007/s11012-019-01022-2

[64] Skalna, I. and Hladík, M., Direct and iterative methods for interval parametric algebraic systems producing parametric solutions. *Numerical Linear Algebra with Applications*, **26** (3), paper e2229, 24 pages (May 2019). https://doi.org/10.1002/nla.2229

[65] Behera, D. and Chakraverty, S., Solving the nondeterministic static governing equations of structures subjected to various forces under fuzzy and interval uncertainty. *International Journal of Approximate Reasoning*, **116**, 43–61 (2020). https://doi.org/10.1016/j.ijar.2019.10.011

[66] Elishakoff, I., Gabriele, S. and Wang, Y., Generalized Galileo Galilei problem in interval setting for functionally related loads. *Archive of Applied Mechanics*, **86** (7), 1203–1217 (2016). https://doi.org/10.1007/s00419-015-1086-4

[67] Popova, E. D., Improved solution to the generalized Galilei's problem with interval loads. *Archive of Applied Mechanics*, **87** (1), 115–127 (2017). https://doi.org/10.1007/s00419-016-1180-2

[68] Popova, E. D., Interval model of equilibrium equations in mechanics. In: Freitag, S., Muhanna, R. L. and Mullen, R. L. (editors), *Proceedings of the 7th International Workshop on Reliable Engineering Computing (REC), Computing with Polymorphic Uncertain Data*, REC 2016, Ruhr University Bochum (Ruhr Universität Bochum, **RU**B), Bochum, Germany, June 15–17, 2016, pp. 241–256. http://www.math.bas.bg/~epopova/papers/16-EPopova-REC16.pdf

[69] Popova, E. D., Interval algebraic approach to equilibrium equations in mechanics. In: Georgiev, K., Todorov, M. and Georgiev, I. (editors), *Advanced Computing in Industrial Mathematics*, Revised Selected Papers of the 10th Annual Meeting of the Bulgarian Section of SIAM, Sofia, Bulgaria, December 21–22, 2015. Book series: *Studies in Computational Intelligence (SCI)*, Vol. 681, pp. 161–173. Springer, Cham, Switzerland, 2017. https://doi.org/10.1007/978-3-319-49544-6_14

[70] Popova, E. D., Equilibrium equations in interval models of structures. *International Journal of Reliability and Safety*, **12** (1/2), 218–235 (2018). https://doi.org/10.1504/IJRS.2018.092525

[71] Popova, E. D., Algebraic solution to interval equilibrium equations of truss structures. *Applied Mathematical Modelling*, **65**, 489–506 (2019). https://doi.org/10.1016/j.apm.2018.08.021

[72] Popova, E. D. and Elishakoff, I., Novel application of interval analysis in strength of materials (extended abstract). Paper presented to 13th Annual Meeting of the Bulgarian Section of SIAM, Sofia, Bulgaria, December 18–20, 2018. Hosted by the Institute of Mechanics, Bulgarian Academy of Sciences. In: *BGSIAM'18, Extended Abstracts*. Fastumprint, Sofia, Bulgaria, 2018, pp. 93–94. http://www.math.bas.bg/IMIdocs/BGSIAM/docs/bgsiam_2018_abstracts.pdf#page=103

[73] Popova, E. D. and Elishakoff, I., Novel interval model applied to derived variables in static and structural problems. *Archive of Applied Mechanics*, **90**, 13 pages (2020; in press; published online 28 November 2019). https://doi.org/10.1007/s00419-019-01644-8

[74] Grandón, C. and Neveu, B., A specific quantifier elimination for inner box test in distance constraints with uncertainties. Research Report RR-5883, INRIA (Institut National de Recherche en Informatique et en Automatique), Sophia Antipolis, France, 2006, 23 pages. Web page at HAL (archives-ouverts.fr): https://hal.inria.fr/inria-00071385/document

[75] Grandón, C. and Goldsztejn, A., Quantifier elimination versus generalized interval evaluation — a comparison on a specific class of quantified constraints. In: Bouchon–Meunier, B., Marsala, Ch., Rifqi, M.

and Yager, R. R. (editors), *Uncertainty and Intelligent Information Systems*. World Scientific, Singapore, 2008, pp. 501–514. https://doi.org/10.1142/9789812792358_0036; web page of the related PDF file (preprint): https://pdfs.semanticscholar.org/c60b/2a4a5d9692ad8cdea4f156c3dc1d7767d488.pdf

[76] Khanh, T. V. and Ogawa, M., SMT for polynomial constraints on real numbers. *Electronic Notes in Theoretical Computer Science*, **289**, 27–40 (2012). https://doi.org/10.1016/j.entcs.2012.11.004

[77] Popova, E. D., Explicit characterization of a class of parametric solution sets. *Comptes Rendus de l'Académie Bulgare des Sciences*, **62** (10), 1207–1216 (2009). Web page of the PDF file of a preprint of this paper (Preprint No. 1/2007, Institute of Mathematics and Informatics) at the Bulgarian Academy of Sciences, Sofia, Bulgaria, 2007: http://www.math.bas.bg/library/images/prepr/1-2007-preprint_ep

[78] Popova, E. D. and Krämer, W., Characterization of *AE* solution sets to a class of parametric linear systems. *Comptes Rendus de l'Académie Bulgare des Sciences*, **64** (3), 325–332 (2011). https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.449.2174&rep=rep1&type=pdf

[79] Skalna, I., *Parametric Interval Algebraic Systems*. Book series: *Studies in Computational Intelligence*, Vol. 766. Springer, Cham, Switzerland, 2018. https://doi.org/10.1007/978-3-319-75187-0

[80] Shary, S. P., Solving the linear interval tolerance problem. *Mathematics and Computers in Simulation*, **39** (1–2), 53–85 (1995). https://doi.org/10.1016/0378-4754(95)00135-K

[81] Shary, S. P., Algebraic solutions to interval linear equations and their applications. In: Alefeld, G. and Herzberger, J. (editors), *Numerical Methods and Error Bounds*, Proceedings of the IMACS–GAMM International Symposium on Numerical Methods and Error Bounds, Oldenburg, Germany, July 9–12, 1995. Book series: *Mathematical Research*, Vol. 89, pp. 224–233. Akademie Verlag, Berlin, 1996. http://www.nsc.ru/interval/shary/Papers/Herz.pdf

[82] Shary, S. P., Algebraic approach to the interval linear static identification, tolerance, and control problems, or One more application of Kaucher arithmetic. *Reliable Computing*, **2** (1), 3–33 (1996). https://doi.org/10.1007/BF02388185

[83] Shary, S. P., Controllable solution set to interval static systems. *Applied Mathematics and Computation*, **86** (2–3), 185–196 (1997). https://doi.org/10.1016/S0096-3003(96)00181-6

[84] Shary, S. P., Outer estimation of generalized solution sets to interval linear systems. *Reliable Computing*, **5**, 323–335 (1999). (Csendes, T., editor, *Developments in Reliable Computing*. Kluwer, Dordrecht, The Netherlands, 1999.) https://doi.org/10.1007/978-94-017-1247-7_25

[85] Beaumont, O. and Philippe, B., Linear interval tolerance problem and linear programming techniques. *Reliable Computing*, **7** (6), 433–447 (2001). https://doi.org/10.1023/A:1014758201565

[86] Goldsztejn, A., A branch and prune algorithm for the approximation of non-linear AE-solution sets. In: *SAC '06, Proceedings of the 2006 21st Annual ACM Symposium on Applied Computing*, Dijon, France, April 23–27, 2006, pp. 1650–1654. ACM (Association for Computing Machinery), New York, 2006. https://doi.org/10.1145/1141277.1141665; web page of the related PDF file at the author's website: http://goldsztejn.com/publications/SAC2006.Goldsztejn.pdf

[87] Goldsztejn, A. and Chabert, G., On the approximation of linear AE-solution sets. Published in the Post-Proceedings of the *12th GAMM–IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics (SCAN 2006)*, 26–29 September 2006, Duisburg, Germany, 8 pages. Published by IEEE (Institute of Electrical and Electronics Engineers), New York, 2006. https://doi.org/10.1109/SCAN.2006.33 (the related PDF file is also available at ResearchGate).

[88] Popova, E. D., Inner estimation of the parametric tolerable solution set. *Computers and Mathematics with Applications*, **66** (9), 1655–1665 (2013). https://doi.org/10.1016/j.camwa.2013.04.007

[89] Popova, E. D. and Hladík, M., Outer enclosures to the parametric *AE* solution set. *Soft Computing*, **17** (8), 1403–1414 (2013). https://doi.org/10.1007/s00500-013-1011-0

[90] Dymova, L., Sevastjanov, P., Pownuk, A. and Kreinovich, V., Practical need for algebraic (equality-type) solutions of interval equations and for extended-zero solutions. In: Wyrzykowski, R., Dongarra, J., Deelman, E. and Karczewski, K. (editors), *Parallel Processing and Applied Mathematics*, Proceedings of PPAM 2017: 12th International Conference on Parallel Processing and Applied Mathematics (PPAM), Lublin, Poland, September 10–13, 2017. Book series: *Lecture Notes in Computer Science (LNCS)*, Vol. 10778; subseries: *Theoretical Computer Science and General Issues (LNTCS)*, Vol. 10778, pp. 412–421. Springer, Cham, Switzerland, 2018. https://doi.org/10.1007/978-3-319-78054-2_39