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# Quantifier-elimination-based interval computations in beam problems studied by using the approximate methods of finite differences and of finite elements

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**Abstract** The rather recent interesting computational method of quantifier elimination already implemented in four computer algebra systems has been already used in many problems of engineering interest including several problems of applied and computational mechanics. Among the previous applications of interest here is mainly the problem of a beam with parametric inequality constraints and under the presence of a loading parameter. This problem was solved by the popular methods (i) of finite differences and (ii) of finite elements in combination with the method of quantifier elimination. Here the same approach is generalized to the case where the loading parameter belongs to an interval. The methods (i) of finite differences and (ii) of finite elements are used again (leading to parametric systems of linear equations) with the computation of the approximate intervals concerning (i) the dimensionless deflection, (ii) the rotation and (iii) the dimensionless bending moment on the whole beam computed on the basis of their values at the nodes used on the beam. In the application of the finite difference method both (i) the purely existential case and (ii) a mixed universal–existential case are considered evidently with respect to the interval loading parameter. The REDLOG computer logic package of the REDUCE computer algebra system has been used again in the present interval computations and the excellent convergence of the obtained approximate intervals computed with the finite difference method is observed. In the purely existential case, up to 3072 intervals on the beam have been successfully used and this is an extremely satisfactory situation in quantifier elimination because it concerns a total number of 3076 variables.

**Keywords** Intervals · Interval arithmetic · Interval variables · Interval parameters · Uncertainty · Ranges · Finite differences · Finite elements · Systems of linear algebraic equations · Beams · Parametric loading · Deflection · Rotation · Bending moment · Quantifiers · Universal quantifier · Existential quantifier · Quantified formulae · Quantified/free variables · Quantifier elimination · Quantifier-free formulae · Symbolic computations · Computer algebra systems · REDLOG

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## 1. Introduction

### 1.1. Symbolic computations, computer algebra systems and quantifier elimination

Computer algebra systems performing both symbolic and numerical computations have proved to constitute an interesting and powerful tool in applied and computational mechanics as well as in a very large number of research fields since the sixties. *ALTRAN* (1965), *REDUCE* (1966), *Mac-syma* (1968), *muMath* (1978), *Maple* (1982), *Derive* (1988) and *Mathematica* (1988) are probably the best known of these systems. An interesting related review concerning applied and structural mechanics was prepared by Pavlović [1] and it was published in 2003. In his own research results, the author used *Derive*, *REDUCE*, *Maple* and *Mathematica*. The present results will be derived by using *REDUCE* and its computer logic package *REDLOG* as will be explained in more detail below.

On the other hand, quantifier elimination in elementary real algebra is a very interesting and algorithmically recent computational method based on symbolic computations and, therefore, on computer algebra systems. The aim of quantifier elimination is simply to eliminate the appearance of the two well-known quantifiers, i.e. (i) the universal quantifier  $\forall$  (for all) and (ii) the existential quantifier  $\exists$  (exists) in formulae including quantified variables (quantified formulae) with at least one of these two quantifiers (e.g.  $\forall x$  or  $\exists x$ ) and to derive completely equivalent (without approximations) formulae from the mathematical and logical points of view. These formulae are called QFFs (quantifier-free formulae) because they are free from these two quantifiers ( $\forall$  and  $\exists$ ) and they include only the free variables in the quantified formula on which quantifier elimination was performed. Additionally, a bibliography on the applications of quantifier elimination in elementary real algebra was prepared by Ratschan [2] in 2012. With respect to the algorithms used in quantifier elimination two of these algorithms are very efficient and extensively used:

- The most popular and general-purpose algorithm for quantifier elimination is CAD (cylindrical algebraic decomposition). This famous algorithm was devised by Collins in 1973 (at first, it was presented at a symposium held at Carnegie-Mellon University) with its first official and complete publication by Collins having appeared in 1975 [3]. The standard book on quantifier elimination and CAD is still the book edited by Caviness and Johnson [4] published (with some delay) in 1998. This book is based on a related symposium held at the Research Institute for Symbolic Computation in Linz, Austria (RISC-Linz) in October 1993 for the celebration of the 20th anniversary of CAD, but it includes all the related fundamental research results going back to the fundamental original results on quantifier elimination by Tarski (during the period 1930–1951) at first officially published in 1948 and, next, in 1951. Moreover, a very large number of interesting research results on CAD by many authors is available in the literature. The best recent implementation of CAD seems to be that by Strzeboński in *Mathematica* [5]. But on the other hand, unfortunately, from the negative point of view it should be mentioned that as was proved by Davenport and Heintz [6] and is now very well known, quantifier elimination for real variables has a doubly-exponential computational complexity [6] and this result is applicable to CAD. Clearly, this result constitutes a serious disadvantage of the method of quantifier elimination and, therefore, a significant obstacle to its wide application especially to quantified formulae with a large total number of variables, i.e. both free and quantified variables. Therefore, here we have been unable to use CAD for the derivation of our results (the QFFs for the beam problem to be considered below), which generally concern a large (or even a very large) total number of quantified variables in the related quantified formulae.
- The second popular method for quantifier elimination is the method of virtual substitution. This method is based on the results mainly by Weispfenning [7–10] and it is the method that will be actually used in this technical report, where only linear algebraic equations in

all the variables are used in the quantified formulae. The Weispfenning method of virtual substitution was preferred here over CAD because of its improved computational complexity for linear algebraic equations (but also for quadratic algebraic equations).

Naturally, from the computational point of view quantifier elimination is not a simple task and, therefore, its implementations in computer algebra systems are rather recent and small in number. The four best known such implementations are

- QEPCAD (now QEPCAD B) based on partial CAD (cylindrical algebraic decomposition) in the SACLIB library mainly by Hong under the guidance of Collins, but with several additional contributors,
- REDLOG mainly based on the method of virtual substitution (and to a less extent on partial CAD) in REDUCE by Dolzmann and Sturm,
- SyNRAC based on the methods of CAD, virtual substitution and Sturm–Habicht sequences mainly by Anai and Yanami,
- The implementation of quantifier elimination in *Mathematica* based on CAD, but also on virtual substitution and several additional algorithms, mainly by Strzeboński.

Although recently we used the implementation of quantifier elimination in *Mathematica* for the derivation of QFFs (quantifier-free formulae), here we found more convenient to use the REDLOG computer logic package of the REDUCE computer algebra system. For the description of REDUCE see its recent (2019) manual by Hearn and Schoepf [11] as well as the books by Rayna [12], Klimov and Rudenko [13], Brackx and Constales [14] and MacCallum and Wright [15].

The REDLOG computer logic package was originally prepared by Dolzmann and Sturm and it is described in its user manual by Dolzmann, Seidl and Sturm [16]; see also its announcement by Dolzmann and Sturm [17] as well as the papers by Dolzmann and Sturm [18], by Sturm [19] and, recently, also by Sturm [20] on the occasion of the completion of thirty years since the discovery of virtual substitution. REDLOG is an extremely well known quantifier elimination package in applications, e.g. in real geometry by Dolzmann, Sturm and Weispfenning [21], in electrical networks by Sturm [22] and in verification and synthesis by Sturm and Tiwari [23].

Following many researchers in various research fields, since 1994 the author has been interested in the application of the method of quantifier elimination to several problems of applied mechanics (see, e.g., Refs. [24–27]). Much more recent applied-mechanics results by the author based on the same computational method, quantifier elimination, can be found in Refs. [28–37]. Additionally, the direct application of CAD to an applied-mechanics problem (optimal solutions to truss problems in structural mechanics) was recently successfully made by Charalampakis and Chatzigiannelis [38].

### 1.2. Interval analysis and its applied-mechanics applications including finite elements/differences

Now with respect to interval analysis we should mention that its modern era began in 1959 with the initiation of publication of the related famous results by Moore and his collaborators; see, e.g., Refs. [39–41]. Of course, undoubtedly, as is very well known, previous interval-related results were also established by several authors. The appearance of these results began in the antiquity with Archimedes for the computation of the number  $\pi$  and reached the much more recent results by Sunaga (in 1958). Additionally, a recent extensive bibliography on interval computations and reliable computing including 784 entries was prepared by Beebe, Kearfott and Kreinovich [42] in 2017.

At this point it should also be mentioned that interval analysis has proved to be an extremely useful tool in applied and computational mechanics during the last thirty years. Among an extremely large number of related very interesting publications see, e.g., the papers (in chronological order, but with the related publications concerning the interval finite element method referenced separately in the next paragraph and also in chronological order) by Dimarogonas [43], Qiu, Chen

and Song [44], Qiu and Elishakoff [45], Kulpa, Pownuk and Skalna [46], Skalna [47], Elishakoff and Ohsaki [48] (book), Elishakoff and Miglis [49, 50], Gabriele and Varano [51], Sofi, Muscolino and Elishakoff [52], Zieniuk, Kapturczak and Kuzelewski [53], Elishakoff, Gabriele and Wang [54], Popova [55], Chakraverty, Hladík and Behera [56], Muscolino, Sofi and Giunta [57], Popova [58, 59], Muscolino and Santoro [60], Faes and Moens [61], Dinh-Cong, Van Hoa and Nguyen-Thoi [62], Behera and Chakraverty [63] and Popova and Elishakoff [64].

Several publications concern the application of interval analysis to the finite element method such as the papers by Dessombz, Thouverez, Laine and Jézéquel [65], Guo and Lü [66], Shao and Su [67], Gao [68], Muhanna, Zhang and Mullen [69], Degrauwe, Lombaert and De Roeck [70], Verhaeghe, Desmet, Vandepitte and Moens [71], Su, Zhu, Wang, Li and Yang [72], Sofi, Romeo, Barrera and Cocks [73], Faes and Moens [74], Ni and Jiang [75], Muhanna and Shahi [76] and Popova [77]. The article in *Wikipedia* on interval finite element [78] is also of interest. Similarly, there are some publications on the application of interval analysis to the finite difference method such as the master thesis by Medina [79] and the paper by Hoffmann, Marciniak and Szyszka [80].

Finally, we can mention that, evidently, applied-mechanics applications (mainly in truss problems in structural mechanics) frequently appear as applications in more mathematical interval-analysis publications concerning parametric interval systems of linear algebraic equations; see, e.g., the recent book by Skalna [81] on parametric interval algebraic systems and the recent papers by Kolev [82], Skalna and Hladík [83] and Popova [84] including applications to truss problems with the book by Skalna [81] also including an application to a steel frame again in structural mechanics.

### *1.3. Relationship between interval analysis and quantifier elimination*

It is well known that quantifiers and quantifier elimination are strongly related to interval analysis. This situation seems to be obvious and natural by taking into consideration the fact that several problems in interval analysis (including the solution sets of parametric or non-parametric interval systems of linear algebraic equations) are expressed in terms of formulae with universally and/or existentially quantified variables. Among a very large number of related results here we can make reference to the results by Grandón and Goldsztejn [85] and Khanh and Ogawa [86]. Additionally, Elishakoff, Gabriele and Wang [54] repeatedly used quantifiers during their study of the generalized Galilei's problem [54], e.g. at the end of Section 2 there [54, p. 1207]. In this way, they have been able to provide a physical meaning to a simple interval equation with a generalized interval.

At this point we can mention that the implementation of quantifier elimination by Strzeboński in *Mathematica* was successfully used by Popova [87] as well as by Popova and Krämer [88] for the characterization of solution sets of parametric interval systems of linear algebraic equations. But, on the other hand, unfortunately, the derived results required too much CPU (central processing unit) time [87] in the computer used or they contained a very large number of logical expressions [88] in comparison with the efficient methods by the same authors for the same computational tasks with respect to parametric interval systems of linear algebraic equations.

In nine recent technical reports [29–37], the author also combined quantifier elimination (by using its implementation in *Mathematica*) with interval analysis in the following problems, almost all of which concern applied mechanics: (i) the computation of ranges of functions that appear in problems of applied mechanics [29], (ii) the determination of ranges of values of stress concentration factors in plane elasticity, more explicitly in notch and hole problems [30], (iii) similarly, for the ranges of values of stress intensity factors at crack tips in plane elasticity problems related to fracture mechanics [31], (iv) the derivation of sharp enclosures of the real roots of the classical parametric quadratic equation but with only one interval coefficient [32], (v) the determination of sharp bounds in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters [33], (vi) the derivation of symbolic intervals in simple problems of applied mechanics [34], (vii) the computation of intervals in three direct and inverse applied mechanics

problems, more explicitly, a classical beam problem, a problem of a beam on a Winkler elastic foundation and the problem of free vibrations of the classical damped harmonic oscillator under critical damping [35], (viii) the determination of intervals (ranges) for the resultants of interval forces satisfying existentially and/or universally quantified formulae [36] and, recently, (ix) the determination of intervals (ranges) in systems of parametric interval linear equilibrium equations in applied mechanics including the case of appearance of both the universal and the existential quantifiers in the quantified formulae concerning the applied mechanics problems under consideration [37].

The present results concern systems of parametric interval linear algebraic equations (resulting by either the finite difference method or the finite element method in a beam problem under a normal triangular loading) with one interval parameter more explicitly with an overall loading interval parameter  $c$  (here  $c \in [1, 2]$ ). As far as systems of parametric interval linear algebraic equations are concerned, these systems were extensively studied in the related literature of interval analysis long ago; see, e.g., the paper by Popova and Hladík [89] as well as the recent book by Skalna [81], which also includes two applied mechanics applications of this class of systems as was already mentioned.

#### 1.4. Contents and aim of this technical report

Beyond the present introductory section, [Section 1](#), the present technical report constituting a generalization of our previous results in Ref. [27] to the case of a parameter (here a dimensionless overall loading parameter  $c$ ) belonging to an interval (interval parameter) is organized as follows:

- In [Section 2](#), we describe the present beam boundary value problem (with a fourth-order ordinary differential equation and four boundary conditions) loaded by a triangular normal loading and under the presence of an interval dimensionless overall loading parameter  $c$  (here  $c \in [1, 2]$ ).
- In [Section 3](#), which is the main section of the present technical report, we exclusively use the method of finite differences. At first, we construct the system of linear algebraic equations based on finite differences for the present boundary value problem continuously using central differences for a greater accuracy of the results. Next, we focus our interest on the computation of the intervals (the ranges) of (i) the dimensionless deflection  $\eta(\xi)$  of the beam, (ii) its rotation (approximately its slope)  $\theta(\xi)$  and (iii) its dimensionless bending moment  $m(\xi)$ . The existence of these intervals is obvious because of the aforementioned assumption of an interval dimensionless overall loading parameter. For this computation, at first, the related quantified formulae are easily constructed in both cases (i) of a resulting purely existentially quantified formula and (ii) of a resulting universally–existentially quantified formula with the universal quantification concerning only the interval dimensionless overall loading parameter  $c$  and, next, quantifier elimination is performed based on the use of the REDLOG computer logic package of the REDUCE computer algebra system. For the sake of convenience of the reader, the related REDLOG–REDUCE sets of commands are also displayed.
- Next, in [Section 4](#), the results of [Section 3](#) are briefly generalized to the case where the finite element method is used instead of the finite difference method, but in this section only with respect to the purely existential case in the quantified formula.
- Finally, [Section 5](#) constitutes a summary of the present results. In this section, some conclusions are also drawn and a related brief discussion is made.

The aim of this technical report is to show the possibility of using the modern computational method of quantifier elimination in combination with the popular methods of finite differences and finite elements in computational mechanics, here in a beam problem, in the case of an overall loading parameter  $c$  belonging to an interval. Here this is successfully achieved by using the REDLOG computer logic package of the classical REDUCE computer algebra system for all three quantities of interest: (i) the dimensionless deflection  $\eta$  of the beam, (ii) the rotation  $\theta$  of the beam and (iii) the dimensionless bending moment  $m$  of the beam and in both cases (i) of a purely existentially and (ii) of a mixed universally–existentially quantified formula with up to 3072 intervals on the beam.



## 2. The beam boundary value problem

For the illustration of the present quantifier-elimination-based approach here we consider the simple problem of a classical beam, i.e. a straight, homogeneous beam of uniform cross-section made of an isotropic elastic material under the validity of the elementary technical theory of beams, i.e. the well-known Euler–Bernoulli theory of beams. The beam has length  $L$  and it lies on the  $Ox$ -axis (with  $x \in [0, L]$ ). The flexural rigidity of the beam is equal to  $EI$ , where  $E$  is the modulus of elasticity (or Young’s modulus) of the isotropic elastic material of the beam and  $I$  is the appropriate moment of inertia of its cross-section. Here this classical beam is assumed simply-supported (through a roller) at its left end  $x = 0$  and fixed (clamped) at its right end  $x = L$ . Therefore, the beam is statically indeterminate. Now as far as the loading of the present beam is concerned, this is assumed to be a triangularly distributed normal loading  $w(x)$  on the beam increasing linearly from its left end  $x = 0$ , where  $w(0) = 0$ , to its right end  $x = L$ , where  $w(L) = w_0$ , i.e. [27, p. 147, Eq. (7)]

$$w(\xi) = w_0 \xi \quad \text{with} \quad \xi := x/L \in [0, 1]. \quad (1)$$

Here the symbol  $\xi$  denotes a dimensionless length variable on the beam with  $\xi \in [0, 1]$ . This beam problem is exactly the beam problem already studied by the author [27] again by using the method of quantifier elimination in its REDLOG implementation in the REDUCE computer algebra system. This simple beam problem was borrowed from the classical book of Beer and Johnston [90, p. 492].

By using the dimensionless variable  $\eta := v/L$  for the deflection  $v = v(\xi)$  of the beam, for the above triangularly distributed normal loading  $w(x)$  or better  $w(\xi)$  in Eq. (1) we have the classical and extremely well-known fourth-order ordinary differential equation [27, p. 147, Eq. (9)]

$$\frac{d^4 \eta}{d\xi^4} = c\xi \quad \text{with} \quad \xi := x/L \in [0, 1], \quad (2)$$

where the parameter  $c$  (overall parameter, constant of the beam) is given by [27, p. 147, Eq. (10)]

$$c = \frac{L^3 w_0}{EI}. \quad (3)$$

Of course, here the present fourth-order ordinary differential equation, Eq. (2), should be supplemented by four appropriate boundary conditions at the ends  $\xi = 0$  and  $\xi = 1$  of the beam. These conditions (using again the dimensionless variables  $\xi$  and  $\eta$ ) have the forms [27, p. 147, Eq. (11)]

$$\eta(0) = 0, \quad \frac{d^2 \eta}{d\xi^2}(0) = 0, \quad \eta(1) = 0, \quad \frac{d\eta}{d\xi}(1) = 0 \quad (4)$$

in the present beam problem. The first, the third and the fourth of these boundary conditions are geometrical (essential) boundary conditions, but the second of them is a mechanical (natural) boundary condition. This condition refers to the vanishing of the bending moment  $M = M(\xi)$ , which is proportional to the second derivative of the deflection of the beam, at the left end  $\xi = 0$  of the beam.

## 3. Application of the method of finite differences

### 3.1. The system of linear algebraic equations

For the approximate solution of the above boundary value problem, Eqs. (2) and (4), at first we will use the classical method of finite differences (here central differences with  $n$  equal intervals of length  $h_0 = L/n$  on the beam with  $x \in [0, L]$  and, equivalently,  $h = 1/n$  with  $\xi := x/L \in [0, 1]$  (see, e.g., the classical book on approximation methods by Zienkiewicz and Morgan [91, Chapter 1, pp. 1–37]) exactly as we did in Ref. [27] for the same beam problem, but without the consideration of intervals for the unknown function  $\eta = \eta(\xi)$  there. Then the ordinary differential equation (2) and the four boundary conditions (4) take the following forms [27, p. 147, Eqs. (12) and (13)]:

$$\frac{\eta_{l+2} - 4\eta_{l+1} + 6\eta_l - 4\eta_{l-1} + \eta_{l-2}}{h^4} = c\xi_l, \quad l = 1, 2, \dots, n-1, \quad \text{with} \quad \eta_l := \eta(\xi_l), \quad (5)$$

and

$$\eta_0 = 0, \quad \eta_1 - 2\eta_0 + \eta_{-1} = 0, \quad \eta_n = 0, \quad \eta_{n+1} - \eta_{n-1} = 0, \quad (6)$$

respectively. In the above equations, we have used the  $n + 3$  equispaced nodes [27, p. 147]

$$\xi_l = lh = \frac{l}{n}, \quad l = -1, 0, 1, \dots, n-1, n, n+1 \quad \text{with} \quad \xi_{-1} < \xi_0 < \xi_1 < \dots < \xi_{n-1} < \xi_n < \xi_{n+1} \quad (7)$$

with the aforementioned constant dimensionless distance (mesh spacing)  $h = 1/n$  between two consecutive nodes  $\xi_l$  and  $\xi_{l+1}$  and the  $n + 3$  symbols  $\eta_l := \eta(\xi_l)$  with  $l = -1, 0, 1, \dots, n-1, n, n+1$ .

The above equations, Eqs. (5) and (6), constitute a system of  $m = n + 3$  linear algebraic equations in  $n + 3$  unknowns. Clearly, these unknowns are the values  $\eta_l$  ( $l = -1, 0, 1, \dots, n-1, n, n+1$ ) of the unknown dimensionless deflection  $\eta = \eta(\xi)$  of the beam, here supplemented by the two additional values  $\eta_{-1}$  and  $\eta_{n+1}$  because of the use of central differences (just for an improved accuracy) in the above finite difference approximations (5) and (6). Therefore, in the present classical beam problem, we derived a system of  $n + 3$  linear algebraic equations in  $n + 3$  unknowns  $\eta_l$  ( $l = -1, 0, 1, \dots, n-1, n, n+1$ ) working exactly as we did in Ref. [27, p. 147, Subsection 3.1].

### 3.2. Quantifier elimination in the purely existential case and the related intervals

Now we are ready to proceed to quantifier elimination in a computational environment with uncertainty assuming that the dimensionless overall parameter  $c$  of the beam (which is essentially a loading parameter) defined in Eq. (3),  $c = L^3 w_0 / (EI)$ , is an interval parameter (a parameter with values in an interval) here with  $c \in [1, 2]$ . Here we are simply interested in the approximate range of the function  $\eta = \eta(\xi)$  on the whole beam ( $0 \leq \xi \leq 1$ ), i.e. we are interested in the approximate interval of this function  $\eta = \eta(\xi)$ , restricting our attention to the presently unknown  $n + 1$  values  $\eta_l$  ( $l = 0, 1, 2, \dots, n-1, n$ ) in the present classical beam problem, i.e. to the approximate values of the dimensionless deflection  $\eta(\xi) = v(\xi)/L$  of the beam, but only at the selected nodes  $\eta_l$ .

To this end we assume that  $\eta(\xi) \in [d_1, d_2]$  on the beam with  $\xi \in [0, 1]$ , but here approximately (approximate values of the endpoints  $d_{1,2}$ ) as was described above. We define the set of variables

$$\mathcal{V}_1 = \{c\} \cup \mathcal{V}_{\eta_1} \quad \text{with} \quad \mathcal{V}_{\eta_1} = \{\eta_{-1}, \eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n, \eta_{n+1}\} \quad (8)$$

as well as our assumptions  $\mathcal{A}_1$  for the overall parameter (or overall constant)  $c$  of the present beam

$$\mathcal{A}_1 = 1 \leq c \leq 2 \quad (9)$$

since we already assumed that  $c \in [1, 2]$ . Having also assumed that  $\eta(\xi) \in [d_1, d_2]$ , we must have

$$d_1 \leq \eta_l \leq d_2 \quad \text{with} \quad l = 0, 1, 2, \dots, n-1, n. \quad (10)$$

These conditions  $\mathcal{C}_\eta$ , which are denoted by the symbol `ycon` in REDUCE, can also be written in a formal conjunctive logical form using the conjunction operator  $\wedge$  (logical ‘and’) as follows:

$$\mathcal{C}_\eta = d_1 \leq \eta_0 \leq d_2 \wedge d_1 \leq \eta_1 \leq d_2 \wedge d_1 \leq \eta_2 \leq d_2 \wedge \dots \wedge d_1 \leq \eta_n \leq d_2. \quad (11)$$

(We observe that the two auxiliary values  $\eta_{-1}$  and  $\eta_{n+1}$  lying outside the beam and serving only computational purposes in the finite difference method do not appear in the above conditions  $\mathcal{C}_\eta$ .) Next, we denote the  $n + 3$  linear finite difference equations (5) and (6) of the present beam problem by the symbol  $\mathcal{E}_1$  (eqs in REDUCE). Then we have the purely existential quantified formula

$$\exists \mathcal{V}_1 \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\eta. \quad (12)$$

In quite a similar manner, we can have conditions  $\mathcal{C}_\theta$  related to the rotation (approximately the slope)  $\theta$  of the beam, of course, here again related to the finite difference method and the selected nodes  $\xi_l = l/n$  on the beam. These conditions  $\mathcal{C}_\theta$  can be written in the conjunctive logical form

$$\mathcal{C}_\theta = \theta_1 \leq \tilde{\theta}_0 \leq \theta_2 \wedge \theta_1 \leq \tilde{\theta}_1 \leq \theta_2 \wedge \theta_1 \leq \tilde{\theta}_2 \leq \theta_2 \wedge \dots \wedge \theta_1 \leq \tilde{\theta}_n \leq \theta_2, \quad (13)$$

where  $\tilde{\theta}_l$  ( $l = 0, 1, 2, \dots, n$ ) denote the approximate nodal values of the rotation  $\theta(\xi)$  of the beam. The related purely existential quantified formula has the form (similar to formula (12))

$$\exists \mathcal{V}_1 \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\theta. \quad (14)$$

Additionally, quite similarly, we can have conditions  $\mathcal{C}_m$  related to the dimensionless bending moment  $m(\xi) = M(\xi)/(EI)$  of the beam (again related to the finite difference method and the nodes  $\eta_l = l/n$  on the beam). These conditions  $\mathcal{C}_m$  can be written in the conjunctive logical form

$$\mathcal{C}_m = m_1 \leq \tilde{m}_0 \leq m_2 \wedge m_1 \leq \tilde{m}_1 \leq m_2 \wedge m_1 \leq \tilde{m}_2 \leq m_2 \wedge \dots \wedge m_1 \leq \tilde{m}_n \leq m_2, \quad (15)$$

where  $\tilde{m}_l$  ( $l = 0, 1, 2, \dots, n$ ) denote the approximate nodal values of the dimensionless bending moment  $m(\xi) = M(\xi)/(EI)$  of the beam. The related purely existential quantified formula has the form (similar to formulae (12) and (14))

$$\exists \mathcal{V}_1 \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_m. \quad (16)$$

Clearly, we can also assume the simultaneous validity of all the conditions  $\mathcal{C}_\eta$ ,  $\mathcal{C}_\theta$  and  $\mathcal{C}_m$ . Then we have the sufficiently more difficult (from the computational point of view) quantified formula

$$\exists \mathcal{V}_1 \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\eta, \mathcal{C}_\theta \text{ and } \mathcal{C}_m. \quad (17)$$

The complete simple set of commands (prepared with *Notepad* in Microsoft Windows 10) related to the present beam problem by using the method of finite differences (as this method was already described in detail) and including the quantifier elimination commands of REDLOG (based on its `rlqe` command) related to all four purely existentially quantified formulae (12) for  $\mathcal{C}_\eta$ , (14) for  $\mathcal{C}_\theta$ , (16) for  $\mathcal{C}_m$  and (17) for  $\mathcal{C}_\eta$ ,  $\mathcal{C}_\theta$  and  $\mathcal{C}_m$  simultaneously, which is the most difficult case, are

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```
load_package "redlog"; rlset ofsf$ operator x,y,th,m,w,eqn; h:=1/n;
for i:=0:n do «x(i):= h*i; w(i):=c*x(i);
th(i):=(y(i+1)-y(i-1))/(2*h);
m(i):=(y(i+1)-2*y(i)+y(i-1))/h^2»$
for i:=1:n-1 do «eqn(i):=(y(i+2)-4*y(i+1)+6*y(i)-4*y(i-1)+y(i-2))/h^4-w(i)»$
eqn(n):=y(0)$ eqn(n+1):=y(n)$ eqn(n+2):=th(n)$ eqn(n+3):=m(0)$
y(-1):=yp$ y(0):=y0$
for i:=0:n+1 do y(i):=mkid(y,i)$
var:={c,yp}$ for i:=0:n+1 do var:=append({y(i)},var)$
eqs:=for i:=1:n+3 mkand eqn(i)=0$
ycon:=for i:=0:n mkand d1<=y(i)<=d2$
thcon:=for i:=0:n mkand th1<=th(i)<=th2$
mcon:=for i:=0:n mkand m1<=m(i)<=m2$
on time; ansy:=rlqe ex(var, eqs and 1<=c<=2 and ycon);
ansth:=rlqe ex(var, eqs and 1<=c<=2 and thcon);
ansm:=rlqe ex(var, eqs and 1<=c<=2 and mcon);
ansall:=rlqe ex(var, eqs and 1<=c<=2 and ycon and thcon and mcon); off time;
soly1:=solve(part(ansy,2,1)=0,d1); soly2:=solve(part(ansy,1,1)=0,d2);
solth1:=solve(part(ansth,2,1)=0,th1); solth2:=solve(part(ansth,1,1)=0,th2);
solm1:=solve(part(ansm,2,1)=0,m1); solm2:=solve(part(ansm,1,1)=0,m2);
on rounded; soly1n:=soly1; soly2n:=soly2;
solth1n:=solth1; solth2n:=solth2;
solm1n:=solm1; solm2n:=solm2; off rounded;
```

---

The approximate results for the intervals (the ranges) of the three functions  $\eta$ ,  $\theta$  and  $m$  were obtained using the present method of finite differences and the quantifier-elimination-based approach



with REDLOG with  $n = 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$  and  $3072$  intervals ( $n + 1$  nodes  $\xi_l$ ) on the beam. These approximate intervals (approximate ranges) of the function  $\eta$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \eta \in \left[0, \frac{1}{192}\right], \quad (18)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \eta \in \left[0, \frac{35}{11264}\right], \quad (19)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \eta \in \left[0, \frac{447}{176128}\right], \quad (20)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \eta \in \left[0, \frac{72611}{29884416}\right], \quad (21)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \eta \in \left[0, \frac{54893433}{22917677056}\right], \quad (22)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \eta \in \left[0, \frac{7001220513}{2932388921344}\right], \quad (23)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \eta \in \left[0, \frac{447711174703}{187655711096832}\right], \quad (24)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \eta \in \left[0, \frac{114585269258439}{48038762529161216}\right], \quad (25)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \eta \in \left[0, \frac{14666324793503019}{6148926419360546816}\right], \quad (26)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow \eta \in \left[0, \frac{1877260032282826867}{787061455778243149824}\right], \quad (27)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow \eta \in \left[0, \frac{30036042351431013599}{12592978788852263026688}\right], \quad (28)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow \eta \in \left[0, \frac{456172060873063050425}{191255852689319792738304}\right], \quad (29)$$

$$\text{theoretical (exact) interval} \Rightarrow \eta \in \left[0, \frac{2\sqrt{5}}{1875}\right] \quad (30)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \eta \in [0, 0.005208333333333], \quad (31)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \eta \in [0, 0.00310724431818], \quad (32)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \eta \in [0, 0.00253792696221], \quad (33)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \eta \in [0, 0.00242972792241], \quad (34)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \eta \in [0, 0.00239524419800], \quad (35)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \eta \in [0, 0.00238754841216], \quad (36)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \eta \in [0, 0.00238581161259], \quad (37)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \eta \in [0, 0.00238526688086], \quad (38)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \eta \in [0, 0.00238518463115], \quad (39)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow \eta \in [0, 0.00238515050953], \quad (40)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow \eta \in [0, 0.00238514197912], \quad (41)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow \eta \in [0, 0.00238514039941], \quad (42)$$

$$\text{theoretical (exact) interval} \Rightarrow \eta \in [0, 0.00238513917600]. \quad (43)$$

Analogous intervals were also obtained for the rotation (approximately the slope)  $\theta$  of the beam with respect to its axis. These approximate intervals (approximate ranges) of the function  $\theta$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \theta \in \left[ -0, \frac{1}{96} \right], \quad (44)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \theta \in \left[ -\frac{35}{5632}, \frac{13}{1408} \right], \quad (45)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \theta \in \left[ -\frac{2289}{352256}, \frac{189}{22016} \right], \quad (46)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \theta \in \left[ -\frac{36869}{5603328}, \frac{2941}{350208} \right], \quad (47)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \theta \in \left[ -\frac{9533573}{1432354816}, \frac{46717}{5595136} \right], \quad (48)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \theta \in \left[ -\frac{610360009}{91637153792}, \frac{746109}{89489408} \right], \quad (49)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \theta \in \left[ -\frac{39091756487}{5864240971776}, \frac{11932285}{1431699456} \right], \quad (50)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \theta \in \left[ -\frac{2501930180993}{375302832259072}, \frac{190894717}{22906667008} \right], \quad (51)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \theta \in \left[ -\frac{160126167571215}{24019243825627136}, \frac{3054228093}{366504574976} \right], \quad (52)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow \theta \in \left[ -\frac{3416060377910103}{512409801938960384}, \frac{48867299965}{5864064811008} \right], \quad (53)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow \theta \in \left[ -\frac{655883827452099785}{98382646787908304896}, \frac{781875401341}{93825003421696} \right], \quad (54)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow \theta \in \left[ -\frac{5603196498900406175}{840479821388612370432}, \frac{35624186177125}{4274901435285504} \right], \quad (55)$$

$$\text{theoretical (exact) interval} \Rightarrow \theta \in \left[ -\frac{1}{150}, \frac{1}{120} \right] \quad (56)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \theta \in [-0.000000000000000, 0.01041666666667], \quad (57)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \theta \in [-0.00621448863636, 0.00923295454545], \quad (58)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \theta \in [-0.00649811500727, 0.00858466569767], \quad (59)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \theta \in [-0.00657983969527, 0.00839786641082], \quad (60)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \theta \in [-0.00665587387532, 0.00834957362967], \quad (61)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \theta \in [-0.00666061726868, 0.00833740010885], \quad (62)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \theta \in [-0.00666612383003, 0.00833435044624], \quad (63)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \theta \in [-0.00666643032224, 0.00833358763775], \quad (64)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \theta \in [-0.00666657821261, 0.00833339691108], \quad (65)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow \theta \in [-0.00666665697062, 0.00833334922787], \quad (66)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow \theta \in [-0.00666666174235, 0.00833333730697], \quad (67)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow \theta \in [-0.00666666391781, 0.00833333509940], \quad (68)$$

$$\text{theoretical (exact) interval} \Rightarrow \theta \in [-0.00666666666667, 0.00833333333333]. \quad (69)$$

Finally, analogous intervals were obtained for the dimensionless bending moment  $m = M/(EI)$  of the present beam. These approximate intervals (approximate ranges) of the function  $m$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow m \in \left[ -\frac{1}{24}, \frac{1}{24} \right], \quad (70)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow m \in \left[ -\frac{23}{704}, \frac{21}{352} \right], \quad (71)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow m \in \left[ -\frac{331}{11008}, \frac{357}{5504} \right], \quad (72)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow m \in \left[ -\frac{21007}{700416}, \frac{5797}{87552} \right], \quad (73)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow m \in \left[ -\frac{667905}{22380544}, \frac{93093}{1398784} \right], \quad (74)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow m \in \left[ -\frac{21348089}{715915264}, \frac{1490853}{22372352} \right], \quad (75)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow m \in \left[ -\frac{227671547}{7635730432}, \frac{23859109}{357924864} \right], \quad (76)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow m \in \left[ -\frac{21854225087}{733013344256}, \frac{381767589}{5726666752} \right], \quad (77)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow m \in \left[ -\frac{699336190841}{23456292798464}, \frac{6108368805}{91626143744} \right], \quad (78)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow m \in \left[ -\frac{22378614036371}{750600295809024}, \frac{97734250405}{1466016202752} \right], \quad (79)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow m \in \left[ -\frac{358057248299543}{12009600437977088}, \frac{1563749404581}{23456250855424} \right], \quad (80)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow m \in \left[ -\frac{16313978513264779}{547187383716544512}, \frac{71248344042701}{1068725358821376} \right], \quad (81)$$

$$\text{theoretical (exact) interval} \Rightarrow m \in \left[ -\frac{\sqrt{5}}{75}, \frac{1}{15} \right] \quad (82)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow m \in [-0.04166666666667, 0.04166666666667], \quad (83)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow m \in [-0.0326704545455, 0.0596590909091], \quad (84)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow m \in [-0.0300690406977, 0.0648619186047], \quad (85)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow m \in [-0.0299921760782, 0.0662120796784], \quad (86)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow m \in [-0.0298431083713, 0.0665528058657], \quad (87)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow m \in [-0.0298192957651, 0.0666381880636], \quad (88)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow m \in [-0.0298166035362, 0.0666595461778], \quad (89)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow m \in [-0.0298142254275, 0.0666648864921], \quad (90)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow m \in [-0.0298144381488, 0.0666662216197], \quad (91)$$

$$\text{for } n = 1 \times 2^{10} = 1024 \quad \Rightarrow m \in [-0.0298142888583, 0.0666665554047], \quad (92)$$

$$\text{for } n = 2 \times 2^{10} = 2048 \quad \Rightarrow m \in [-0.0298142515356, 0.0666666388512], \quad (93)$$

$$\text{for } n = 3 \times 2^{10} = 3072 \quad \Rightarrow m \in [-0.0298142446240, 0.0666666543042], \quad (94)$$

$$\text{theoretical (exact) interval} \Rightarrow m \in [-0.0298142397000, 0.0666666666667]. \quad (95)$$

Now, as far as the required computational times for quantifier elimination for these computations with REDLOG are concerned, these times,  $t_\eta$  for the dimensionless deflection  $\eta = v/L$  of the beam,  $t_\theta$  for the rotation (approximately the slope)  $\theta$  of the beam,  $t_m$  for the dimensionless bending moment  $m = M/(EI)$  of the beam and  $t_{\eta\theta m}$  for all three functions  $\eta$ ,  $\theta$  and  $m$  (but with a single quantifier elimination command, not simply the sum of the times  $t_\eta$ ,  $t_\theta$  and  $t_m$ ), are (in s, seconds)

$$\text{for } n = 2 \Rightarrow t_\eta = 0.016, \quad t_\theta = 0.000, \quad t_m = 0.000, \quad t_{\eta\theta m} = 0.000, \quad (96)$$

$$\text{for } n = 4 \Rightarrow t_\eta = 0.016, \quad t_\theta = 0.000, \quad t_m = 0.000, \quad t_{\eta\theta m} = 0.015, \quad (97)$$

$$\text{for } n = 8 \Rightarrow t_\eta = 0.015, \quad t_\theta = 0.000, \quad t_m = 0.016, \quad t_{\eta\theta m} = 0.015, \quad (98)$$

$$\text{for } n = 16 \Rightarrow t_\eta = 0.015, \quad t_\theta = 0.031, \quad t_m = 0.016, \quad t_{\eta\theta m} = 0.031, \quad (99)$$

$$\text{for } n = 32 \Rightarrow t_\eta = 0.047, \quad t_\theta = 0.047, \quad t_m = 0.047, \quad t_{\eta\theta m} = 0.140, \quad (100)$$

$$\text{for } n = 64 \Rightarrow t_\eta = 0.187, \quad t_\theta = 0.220, \quad t_m = 0.201, \quad t_{\eta\theta m} = 0.579, \quad (101)$$

$$\text{for } n = 128 \Rightarrow t_\eta = 0.890, \quad t_\theta = 0.923, \quad t_m = 0.875, \quad t_{\eta\theta m} = 3.376, \quad (102)$$

$$\text{for } n = 256 \Rightarrow t_\eta = 4.548, \quad t_\theta = 4.670, \quad t_m = 4.905, \quad t_{\eta\theta m} = 24.111, \quad (103)$$

$$\text{for } n = 512 \Rightarrow t_\eta = 32.628, \quad t_\theta = 32.252, \quad t_m = 34.000, \quad t_{\eta\theta m} = 171.907, \quad (104)$$

$$\text{for } n = 1024 \Rightarrow t_\eta = 233.109, \quad t_\theta = 239.613, \quad t_m = 255.095, \quad t_{\eta\theta m} = 1249.213, \quad (105)$$

$$\text{for } n = 2048 \Rightarrow t_\eta = 1677.460, \quad t_\theta = 1753.390, \quad t_m = 1876.713, \quad t_{\eta\theta m} = \text{not available}, \quad (106)$$

$$\text{for } n = 3072 \Rightarrow t_\eta = 5648.303, \quad t_\theta = 6083.654, \quad t_m = 6767.331, \quad t_{\eta\theta m} = \text{not available}. \quad (107)$$

All the above times do not include the corresponding times for garbage collection in REDUCE, which, of course, are sufficiently smaller. These garbage collection times,  $\tau_\eta$  for the dimensionless deflection  $\eta = v/L$ ,  $\tau_\theta$  for the rotation (approximately the slope)  $\theta$ ,  $\tau_m$  for the dimensionless bending moment  $m = M/(EI)$  and  $\tau_{\eta\theta m}$  for all three functions  $\eta$ ,  $\theta$  and  $m$ , are (again in s, seconds)

$$\text{for } n = 2 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (108)$$

$$\text{for } n = 4 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (109)$$

$$\text{for } n = 8 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (110)$$

$$\text{for } n = 16 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.047, \quad (111)$$

$$\text{for } n = 32 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.031, \quad \tau_{\eta\theta m} = 0.032, \quad (112)$$

$$\text{for } n = 64 \Rightarrow \tau_\eta = 0.047, \quad \tau_\theta = 0.015, \quad \tau_m = 0.032, \quad \tau_{\eta\theta m} = 0.093, \quad (113)$$

$$\text{for } n = 128 \Rightarrow \tau_\eta = 0.157, \quad \tau_\theta = 0.155, \quad \tau_m = 0.156, \quad \tau_{\eta\theta m} = 0.373, \quad (114)$$

$$\text{for } n = 256 \Rightarrow \tau_\eta = 0.563, \quad \tau_\theta = 0.594, \quad \tau_m = 0.657, \quad \tau_{\eta\theta m} = 1.641, \quad (115)$$

$$\text{for } n = 512 \Rightarrow \tau_\eta = 2.657, \quad \tau_\theta = 2.612, \quad \tau_m = 2.827, \quad \tau_{\eta\theta m} = 6.384, \quad (116)$$

$$\text{for } n = 1024 \Rightarrow \tau_\eta = 10.049, \quad \tau_\theta = 9.991, \quad \tau_m = 10.858, \quad \tau_{\eta\theta m} = 33.756, \quad (117)$$

$$\text{for } n = 2048 \Rightarrow \tau_\eta = 53.126, \quad \tau_\theta = 45.845, \quad \tau_m = 57.125, \quad \tau_{\eta\theta m} = \text{not available}, \quad (118)$$

$$\text{for } n = 3072 \Rightarrow \tau_\eta = 139.367, \quad \tau_\theta = 145.856, \quad \tau_m = 157.184, \quad \tau_{\eta\theta m} = \text{not available}. \quad (119)$$

Naturally, all these times are approximate and they vary when the same quantifier elimination computations are repeated with the REDLOG package of REDUCE as was really observed. Moreover, evidently, they depend on the computer used, here a personal computer with an Intel Core i5 CPU 650 at 3.20 GHz and 4 GB of RAM under the popular 64-bit Windows 10 operating system. Additionally, obviously, we observe that the  $\tau$ -times required for garbage collection are seen to be much smaller than the corresponding  $t$ -times required for the actual quantifier elimination computations. Finally, we should mention that we did not perform quantifier elimination in the difficult case of the simultaneous computation of the three intervals (ranges) of  $\eta$ ,  $\theta$  and  $m$  for  $n = 2048$

and  $n = 3072$  because of the required excessive computational times  $t_{\eta\theta m}$ . Analogously, we did not proceed to the computation of the intervals of the functions  $\eta = v/L$ ,  $\theta$  and  $m = M/(EI)$  on the beam for higher values of the number  $n$  of intervals ( $n > 3072$ ) used in the finite difference method.

### 3.3. Quantifier elimination in a mixed universal–existential case and the related intervals

In the previous subsection, we considered the purely existential case in the quantified formulae with the overall (mainly loading) parameter  $c = L^3 w_0/(EI)$  in Eq. (3) being an interval parameter (with  $c \in [1, 2]$ ) and the unknowns  $\eta_l$  computed by using the method of finite differences and the related system of linear algebraic equations. In this subsection, we consider a somewhat different case. This is the mixed universal–existential case in the quantified formulae, where now the overall (mainly loading) parameter  $c$  is universally quantified whereas the unknowns  $\eta_l$  in the aforementioned system remain existentially quantified. We again use the REDLOG computer logic package of the REDUCE computer algebra system, but now the computational task seems to be more difficult for REDLOG and REDUCE and this happens because of the presence in the quantified formulae of both quantifiers  $\forall$  (for all) with respect to  $c$  and  $\exists$  (exists) with respect to the unknowns  $\eta_l$  in the finite difference linear equations. Here we assume again that the interval overall parameter  $c$  lies in the interval  $[1, 2]$  exactly as previously. Under these conditions we have been able to work with only up to  $n = 512$  intervals on the beam during the application of the method of finite differences.

The interval variables  $\mathcal{V}_{\eta_1}$  are given again by the second of Eqs. (8), whereas the interval variables  $\mathcal{V}_1$  in the first of Eqs. (8) are not useful any more. Next, the assumptions  $\mathcal{A}_1 = 1 \leq c \leq 2$  in Eq. (9) still hold true as well as the conditions  $\mathcal{C}_\eta$  if we are interested in the range of the dimensionless deflection  $\eta := v/L$  or  $\mathcal{C}_\theta$  if we are interested in the range of the rotation  $\theta$  or  $\mathcal{C}_m$  if we are interested in the range of the dimensionless bending moment  $m := M/(EI)$ . But the related quantified formulae (12), (14), (16) and (17) now take the slightly modified corresponding forms

$$\forall c \exists \mathcal{V}_{\eta_1} \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\eta, \quad (120)$$

$$\forall c \exists \mathcal{V}_{\eta_1} \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\theta, \quad (121)$$

$$\forall c \exists \mathcal{V}_{\eta_1} \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_m, \quad (122)$$

$$\forall c \exists \mathcal{V}_{\eta_1} \text{ such that Eqs. } \mathcal{E}_1 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\eta, \mathcal{C}_\theta, \mathcal{C}_m. \quad (123)$$

The related simple set of commands in REDUCE and REDLOG is similar to that used in the previous subsection, but now the quantifier elimination commands (again based on the `rlqe` command of REDLOG) include both quantifiers  $\forall$  (`all` in REDLOG) and  $\exists$  (`ex` in REDLOG) as follows:

---

```
ansy:=rlqe all(c, not(1<=c<=2) or ex(var, eqs and ycon));
ansth:=rlqe all(c, not(1<=c<=2) or ex(var, eqs and thcon));
ansm:=rlqe all(c, not(1<=c<=2) or ex(var, eqs and mcon));
ansall:=rlqe all(c, not(1<=c<=2) or ex(var, eqs and ycon and thcon and mcon));
```

---

with the symbol `var` now referring to the  $n + 3$  variables  $V_{\eta_1}$  in the second of Eqs. (8). These four quantifier elimination commands correspond to the above four universally–existentially quantified formulae (120) for  $\mathcal{C}_\eta$ , (121) for  $\mathcal{C}_\theta$ , (122) for  $\mathcal{C}_m$  and (123) for  $\mathcal{C}_\eta$ ,  $\mathcal{C}_\theta$  and  $\mathcal{C}_m$  simultaneously, respectively, with the last case, obviously, computationally being again the most difficult one.

Alternatively, because of the well-known logical equivalence

$$A \Rightarrow B \equiv \neg A \vee B, \quad (124)$$

we can use the following equivalent and probably preferable REDLOG commands:

---

```
ansy:=rlqe all(c, (1<=c<=2) impl ex(var, eqs and ycon));
ansth:=rlqe all(c, (1<=c<=2) impl ex(var, eqs and thcon));
```



```
ansm:=rlqe all(c, (1<=c<=2) impl ex(var, eqs and mcon));
ansall:=rlqe all(c, (1<=c<=2) impl ex(var, eqs and ycon and thcon and mcon));
```

respectively. The above four slightly simpler commands are based on the  $\Rightarrow$  (impl in REDLOG) implication operator instead of the previous related commands, which were based on the  $\neg$  (not in REDLOG) negation operator and, simultaneously, on the  $\vee$  (or in REDLOG) disjunction operator.

Here the approximate results for the intervals (the ranges) of the three functions  $\eta = v/L$ ,  $\theta$  and  $m = M/(EI)$  of the beam were again computed using the method of finite differences and the quantifier-elimination-based approach with REDLOG with  $n = 2, 4, 8, 16, 32, 64, 128, 256$  and  $512$  intervals ( $n + 1$  nodes  $\xi_l$ ) on the beam. (It should be mentioned that in the present application with universally–existentially quantified formulae, we were unable to proceed to  $n = 1024$  or more intervals on the beam.) The derived approximate intervals (approximate ranges) of the function  $\eta$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \quad \eta \in \left[0, \frac{1}{96}\right], \quad (125)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \quad \eta \in \left[0, \frac{35}{5632}\right], \quad (126)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \quad \eta \in \left[0, \frac{447}{88064}\right], \quad (127)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \quad \eta \in \left[0, \frac{72611}{14942208}\right], \quad (128)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \quad \eta \in \left[0, \frac{54893433}{11458838528}\right], \quad (129)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \quad \eta \in \left[0, \frac{7001220513}{1466194460672}\right], \quad (130)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \quad \eta \in \left[0, \frac{447711174703}{93827855548416}\right], \quad (131)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \quad \eta \in \left[0, \frac{114585269258439}{24019381264580608}\right], \quad (132)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \quad \eta \in \left[0, \frac{14666324793503019}{3074463209680273408}\right], \quad (133)$$

$$\text{theoretical (exact) interval} \Rightarrow \eta \in \left[0, \frac{4\sqrt{5}}{1875}\right] \quad (134)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \quad \eta \in [0, 0.0104166666667], \quad (135)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \quad \eta \in [0, 0.00621448863636], \quad (136)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \quad \eta \in [0, 0.00507585392442], \quad (137)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \quad \eta \in [0, 0.00485945584481], \quad (138)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \quad \eta \in [0, 0.00479048839600], \quad (139)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \quad \eta \in [0, 0.00477509682433], \quad (140)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \quad \eta \in [0, 0.00477162322517], \quad (141)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \quad \eta \in [0, 0.00477053376173], \quad (142)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \quad \eta \in [0, 0.00477036926229], \quad (143)$$

$$\text{theoretical (exact) interval} \Rightarrow \eta \in [0, 0.00477027835200]. \quad (144)$$

At this point we observe that the above results (both exact and numerical) for the approximate interval (range) of the dimensionless deflection  $\eta = v/L$  of the present beam (obtained by the method of finite differences) in the present case of universal quantification of the overall parameter  $c$  ( $\forall c$  in the quantified formula (120)) coincide with the previous results for the same deflection  $\eta$  in the case of existential quantification of the same parameter  $c$  simply after the multiplication of the corresponding endpoints by the number 2. This is obvious if we take into account the present assumptions  $\mathcal{A}_1 = 1 \leq c \leq 2$  in Eq. (9) and the fact that now (with the quantifier  $\forall$ ) the interval parameter  $c$  “moved” from the left endpoint  $c_1 = 1$  of the interval  $[1, 2]$  (with the quantifier  $\exists$ ) to the right endpoint  $c_2 = 2$  of the same interval (with the quantifier  $\forall$ ). Hence, under the present assumptions  $\mathcal{A}_1$  the lengths of the above approximate intervals in the universal case ( $\forall c$ ) have been doubled in comparison with the corresponding lengths in the previous, the existential case ( $\exists c$ ).

Analogous intervals were also obtained for the rotation (approximately the slope)  $\theta$  of the present beam with respect to its axis  $Ox$ . These approximate intervals (approximate ranges) of  $\theta$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \theta \in \left[ -0, \frac{1}{48} \right], \quad (145)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \theta \in \left[ -\frac{35}{2816}, \frac{13}{704} \right], \quad (146)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \theta \in \left[ -\frac{2289}{176128}, \frac{189}{11008} \right], \quad (147)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \theta \in \left[ -\frac{36869}{2801664}, \frac{2941}{175104} \right], \quad (148)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \theta \in \left[ -\frac{9533573}{716177408}, \frac{46717}{2797568} \right], \quad (149)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \theta \in \left[ -\frac{610360009}{45818576896}, \frac{746109}{44744704} \right], \quad (150)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \theta \in \left[ -\frac{39091756487}{2932120485888}, \frac{11932285}{715849728} \right], \quad (151)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \theta \in \left[ -\frac{2501930180993}{187651416129536}, \frac{190894717}{11453333504} \right], \quad (152)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \theta \in \left[ -\frac{160126167571215}{12009621912813568}, \frac{3054228093}{183252287488} \right], \quad (153)$$

$$\text{theoretical (exact) interval} \Rightarrow \theta \in \left[ -\frac{1}{75}, \frac{1}{60} \right] \quad (154)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow \theta \in [-0.000000000000, 0.0208333333333], \quad (155)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow \theta \in [-0.0124289772727, 0.0184659090909], \quad (156)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow \theta \in [-0.0129962300145, 0.0171693313953], \quad (157)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow \theta \in [-0.0131596793905, 0.0167957328216], \quad (158)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow \theta \in [-0.0133117477506, 0.0166991472593], \quad (159)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow \theta \in [-0.0133212345374, 0.0166748002177], \quad (160)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow \theta \in [-0.0133322476601, 0.0166687008925], \quad (161)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow \theta \in [-0.0133328606445, 0.0166671752755], \quad (162)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow \theta \in [-0.0133331564252, 0.0166667938222], \quad (163)$$

$$\text{theoretical (exact) interval} \Rightarrow \theta \in [-0.0133333333333, 0.0166666666667]. \quad (164)$$

At this point we observe again that the above results (both exact and numerical) for the approximate interval (range) of the rotation (approximately the slope)  $\theta$  of the beam under consideration in the present case of universal quantification of the overall parameter  $c$  ( $\forall c$  in the quantified formula (121)) coincide with the previous results for the same rotation  $\theta$  in the case of existential quantification of the same overall parameter  $c$  simply after the multiplication of the corresponding endpoints by the number 2. This situation is completely analogous to that already observed in the previous case concerning the dimensionless deflection  $\eta = v/L$  of the same beam.

Finally, analogous intervals were obtained for the dimensionless bending moment  $m = M/(EI)$  of the present beam. These approximate intervals (approximate ranges) of the function  $m$  are

$$\text{for } n = 2^1 = 2 \quad \Rightarrow m \in \left[ -\frac{1}{12}, \frac{1}{12} \right], \quad (165)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow m \in \left[ -\frac{23}{352}, \frac{21}{176} \right], \quad (166)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow m \in \left[ -\frac{331}{5504}, \frac{357}{2752} \right], \quad (167)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow m \in \left[ -\frac{21007}{350208}, \frac{5797}{43776} \right], \quad (168)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow m \in \left[ -\frac{667905}{11190272}, \frac{93093}{699392} \right], \quad (169)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow m \in \left[ -\frac{21348089}{357957632}, \frac{1490853}{11186176} \right], \quad (170)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow m \in \left[ -\frac{227671547}{3817865216}, \frac{23859109}{178962432} \right], \quad (171)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow m \in \left[ -\frac{21854225087}{366506672128}, \frac{381767589}{2863333376} \right], \quad (172)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow m \in \left[ -\frac{699336190841}{11728146399232}, \frac{6108368805}{45813071872} \right], \quad (173)$$

$$\text{theoretical (exact) interval} \Rightarrow m \in \left[ -\frac{2\sqrt{5}}{75}, \frac{2}{15} \right] \quad (174)$$

and numerically (with additional, but not significant, approximations, rounding approximations)

$$\text{for } n = 2^1 = 2 \quad \Rightarrow m \in [-0.08333333333333, 0.08333333333333], \quad (175)$$

$$\text{for } n = 2^2 = 4 \quad \Rightarrow m \in [-0.0653409090909, 0.119318181818], \quad (176)$$

$$\text{for } n = 2^3 = 8 \quad \Rightarrow m \in [-0.0601380813953, 0.129723837209], \quad (177)$$

$$\text{for } n = 2^4 = 16 \quad \Rightarrow m \in [-0.0599843521564, 0.132424159357], \quad (178)$$

$$\text{for } n = 2^5 = 32 \quad \Rightarrow m \in [-0.0596862167425, 0.133105611731], \quad (179)$$

$$\text{for } n = 2^6 = 64 \quad \Rightarrow m \in [-0.0596385915303, 0.133276376127], \quad (180)$$

$$\text{for } n = 2^7 = 128 \quad \Rightarrow m \in [-0.0596332070723, 0.133319092356], \quad (181)$$

$$\text{for } n = 2^8 = 256 \quad \Rightarrow m \in [-0.0596284508550, 0.133329772984], \quad (182)$$

$$\text{for } n = 2^9 = 512 \quad \Rightarrow m \in [-0.0596288762977, 0.133332443239], \quad (183)$$

$$\text{theoretical (exact) interval} \Rightarrow m \in [-0.0596284794000, 0.133333333333]. \quad (184)$$

At this point we observe again that the above results (both exact and numerical) for the approximate interval (range) of the dimensionless bending moment  $m = M/(EI)$  of the present beam

in the present case of universal quantification of the overall parameter  $c$  ( $\forall c$  in the quantified formula (122)) coincide with the previous results for the same moment  $m$  in the case of existential quantification of the same parameter  $c$  simply after the multiplication of the corresponding endpoints by the number 2. Clearly, this situation is completely analogous to that observed in the previous two cases concerning the dimensionless deflection  $\eta = v/L$  and the rotation  $\theta$  of the beam.

Now, as far as the required computational times for quantifier elimination for these computations with REDLOG are concerned, these times,  $t_\eta$  for the dimensionless deflection  $\eta = v/L$  of the beam,  $t_\theta$  for the rotation (approximately the slope)  $\theta$  of the beam,  $t_m$  for the dimensionless bending moment  $m = M/(EI)$  of the beam and  $t_{\eta\theta m}$  for all three quantities  $\eta$ ,  $\theta$  and  $m$  (but with the same quantifier elimination command, not simply the sum of the times  $t_\eta$ ,  $t_\theta$  and  $t_m$ ), are (in s, seconds)

$$\text{for } n = 2 \Rightarrow t_\eta = 0.011, \quad t_\theta = 0.005, \quad t_m = 0.004, \quad t_{\eta\theta m} = 0.000, \quad (185)$$

$$\text{for } n = 4 \Rightarrow t_\eta = 0.008, \quad t_\theta = 0.009, \quad t_m = 0.000, \quad t_{\eta\theta m} = 0.024, \quad (186)$$

$$\text{for } n = 8 \Rightarrow t_\eta = 0.016, \quad t_\theta = 0.015, \quad t_m = 0.005, \quad t_{\eta\theta m} = 0.043, \quad (187)$$

$$\text{for } n = 16 \Rightarrow t_\eta = 0.048, \quad t_\theta = 0.036, \quad t_m = 0.055, \quad t_{\eta\theta m} = 0.131, \quad (188)$$

$$\text{for } n = 32 \Rightarrow t_\eta = 0.187, \quad t_\theta = 0.208, \quad t_m = 0.213, \quad t_{\eta\theta m} = 0.624, \quad (189)$$

$$\text{for } n = 64 \Rightarrow t_\eta = 1.307, \quad t_\theta = 1.317, \quad t_m = 1.336, \quad t_{\eta\theta m} = 5.592, \quad (190)$$

$$\text{for } n = 128 \Rightarrow t_\eta = 12.219, \quad t_\theta = 12.474, \quad t_m = 12.841, \quad t_{\eta\theta m} = 70.864, \quad (191)$$

$$\text{for } n = 256 \Rightarrow t_\eta = 151.405, \quad t_\theta = 157.161, \quad t_m = 158.350, \quad t_{\eta\theta m} = 1183.510, \quad (192)$$

$$\text{for } n = 512 \Rightarrow t_\eta = 2376.813, \quad t_\theta = 2491.581, \quad t_m = 2439.824, \quad t_{\eta\theta m} = \text{not available.} \quad (193)$$

The above times do not include the corresponding times for garbage collection in REDUCE, which, of course, are sufficiently smaller. These garbage collection times, i.e.  $\tau_\eta$  for the dimensionless deflection  $\eta = v/L$ ,  $\tau_\theta$  for the rotation (approximately the slope)  $\theta$ ,  $\tau_m$  for the dimensionless bending moment  $m = M/(EI)$  and  $\tau_{\eta\theta m}$  for all three quantities  $\eta$ ,  $\theta$  and  $m$  are (again in s, seconds)

$$\text{for } n = 2 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (194)$$

$$\text{for } n = 4 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (195)$$

$$\text{for } n = 8 \Rightarrow \tau_\eta = 0.000, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.000, \quad (196)$$

$$\text{for } n = 16 \Rightarrow \tau_\eta = 0.043, \quad \tau_\theta = 0.000, \quad \tau_m = 0.000, \quad \tau_{\eta\theta m} = 0.016, \quad (197)$$

$$\text{for } n = 32 \Rightarrow \tau_\eta = 0.034, \quad \tau_\theta = 0.030, \quad \tau_m = 0.031, \quad \tau_{\eta\theta m} = 0.074, \quad (198)$$

$$\text{for } n = 64 \Rightarrow \tau_\eta = 0.133, \quad \tau_\theta = 0.153, \quad \tau_m = 0.162, \quad \tau_{\eta\theta m} = 0.313, \quad (199)$$

$$\text{for } n = 128 \Rightarrow \tau_\eta = 0.606, \quad \tau_\theta = 0.676, \quad \tau_m = 0.735, \quad \tau_{\eta\theta m} = 1.740, \quad (200)$$

$$\text{for } n = 256 \Rightarrow \tau_\eta = 5.263, \quad \tau_\theta = 3.662, \quad \tau_m = 3.694, \quad \tau_{\eta\theta m} = 10.624, \quad (201)$$

$$\text{for } n = 512 \Rightarrow \tau_\eta = 42.124, \quad \tau_\theta = 46.510, \quad \tau_m = 50.133, \quad \tau_{\eta\theta m} = \text{not available.} \quad (202)$$

Naturally, exactly as in the previous subsection, all these times are approximate and vary simply if the same quantifier elimination computations are repeated with the REDLOG computer logic package of REDUCE as was really observed. Moreover, evidently, they depend again on the computer used as has been already mentioned. Additionally, obviously, we observe again that the  $\tau$ -times required for garbage collection in REDUCE are directly seen to be much smaller than the corresponding  $t$ -times required for the actual quantifier elimination computations. We should also mention that for  $n = 512$  we did not perform quantifier elimination in the difficult case of the simultaneous computation of the three intervals for  $\eta = v/L$ ,  $\theta$  and  $m = M/(EI)$  because of the required excessive computational time. Finally, analogously to the previous subsection, we did not proceed to the computation of the intervals (the ranges) of the functions  $\eta = v/L$ ,  $\theta$  and  $m = M/(EI)$  of the beam for higher values of the number  $n$  of intervals ( $n > 512$ ) used in the finite difference method.

#### 4. Application of the method of finite elements

Quite similarly, we can work with the method of finite elements (see, e.g., the monographs by Zienkiewicz, Taylor and Zhu [92], Reddy [93] and Bathe [94]) for the present beam problem as a generalization of the related results by the method of finite elements obtained in Ref. [27, Section 4, pp. 150–152] in the case of inequality constraints instead of intervals here. Evidently, in this section, the essential change in comparison with the results of the previous section is the substitution of the (approximate) system of linear equations resulting in the method of finite differences by the corresponding (and also approximate) system of linear equations resulting in the method of finite elements. On the other hand, the computational approach based on quantifier elimination performed on the related existentially quantified formulae by using the REDLOG computer logic package of the REDUCE computer algebra system remains essentially the same. The present computational approach leads to the derivation of the approximate intervals of the dimensionless deflection  $\eta = \eta(\xi)$  (with  $\xi := x/L \in [0, 1]$ ) of the beam and of the related rotation  $\theta(\xi)$  on the beam based on the approximate values of these quantities at the nodes used in the finite element method.

At this point it should be mentioned that Beltzer [95, Chapter F, pp. 77–192] describes in detail the use of the computer algebra system *Maple* in the application of the finite element method to beam and several additional applied-mechanics problems. Here we will also use a computer algebra system (REDUCE instead of *Maple* having been used by Beltzer [95]), but now with respect to the computation of intervals under uncertainty conditions here with an interval parameter  $c$  mainly concerning the distributed normal loading  $w(\xi) = w_0 \xi$  in Eq. (1) (with  $\xi := x/L \in [0, 1]$ ) of the beam. The details of the finite element computations can be found in a large number of books including the monographs by Zienkiewicz, Taylor and Zhu [92], Reddy [93], Bathe [94] and Beltzer [95, 96].

More explicitly, if we adopt the classical Hermite cubic interpolation polynomial on a beam element (with nodes its endpoints), which has the form [95, p. 78]

$$v(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 \quad \text{here with } \xi := x/L \in [0, 1], \quad (203)$$

then the related stiffness matrix  $\mathbf{K}^{(e)}$  of each element of the beam is found to be given by [95, p. 80]

$$\mathbf{K}^{(e)} = \begin{bmatrix} 12/h^3 & 6/h^2 & -12/h^3 & 6/h^2 \\ 6/h^2 & 4/h & -6/h^2 & 2/h \\ -12/h^3 & -6/h^2 & 12/h^3 & -6/h^2 \\ 6/h^2 & 2/h & -6/h^2 & 4/h \end{bmatrix}. \quad (204)$$

This stiffness matrix is written here in a dimensionless form without the appearance of the length  $L$  and the flexural rigidity  $EI$  of the beam and with  $h = 1/n$  exactly as in the previous section in the method of finite differences there. On the basis of the above dimensionless stiffness matrix  $\mathbf{K}^{(e)}$ , the assembled dimensionless stiffness matrix  $\mathbf{K}$  (for all  $n$  finite elements of the beam) can also be directly constructed as is described by Beltzer [95, Section F14, pp. 117–121]. For example, in the case of  $n = 4$  finite elements, the assembled dimensionless stiffness matrix  $\mathbf{K}$  has the form

$$\mathbf{K} = \begin{bmatrix} 768 & 96 & -768 & 96 & 0 & 0 & 0 & 0 & 0 & 0 \\ 96 & 16 & -96 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & -96 & 1536 & 0 & -768 & 96 & 0 & 0 & 0 & 0 \\ 96 & 8 & 0 & 32 & -96 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & -768 & -96 & 1536 & 0 & -768 & 96 & 0 & 0 \\ 0 & 0 & 96 & 8 & 0 & 32 & -96 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -768 & -96 & 1536 & 0 & -768 & 96 \\ 0 & 0 & 0 & 0 & 96 & 8 & 0 & 32 & -96 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -768 & -96 & 768 & -96 \\ 0 & 0 & 0 & 0 & 0 & 0 & 96 & 8 & -96 & 16 \end{bmatrix}. \quad (205)$$



Furthermore, the shape functions (or interpolation functions)  $d_k(\xi)$  (with  $k = 1, 2, 3, 4$ ) on each finite element should be taken into account as well. These functions are also very well known in beam problems and they can be seen to be given by [95, p. 81] (here again in dimensionless form, i.e. with  $\xi := x/L$  and  $h = 1/n$ )

$$d_1(\xi) = 1 - 3\frac{\xi^2}{h^2} + 2\frac{\xi^3}{h^3}, \quad (206)$$

$$d_2(\xi) = \xi - 2\frac{\xi^2}{h} + \frac{\xi^3}{h^2}, \quad (207)$$

$$d_3(\xi) = 3\frac{\xi^2}{h^2} - 2\frac{\xi^3}{h^3}, \quad (208)$$

$$d_4(\xi) = -\frac{\xi^2}{h} + \frac{\xi^3}{h^2} \quad (209)$$

with  $\xi \in [0, 1]$ . Next, the integration of the functions  $d_k(\xi)w^*(\xi + jh)$  on the interval  $[0, 1]$  (here with  $k = 1, 2, 3, 4$  and  $w^*(\xi) = c\xi$  with  $\xi \in [0, 1]$  denoting the dimensionless normal loading of the beam) on the elements of the beam ( $j = 0, 1, 2, \dots, n - 1$ ) permits the direct computation of the equivalent (to  $w^*(\xi) = c\xi$ ) generalized nodal force vector  $\mathbf{Q}$ .

For example, for  $n = 4$  finite elements on the beam we have the following integral formulae and the related results:

$$Q_1 = \int_0^h d_1(\xi)w^*(\xi) d\xi = \frac{3c}{320}, \quad (210)$$

$$Q_2 = \int_0^h d_2(\xi)w^*(\xi) d\xi = \frac{c}{1920}, \quad (211)$$

$$Q_3 = \int_0^h d_3(\xi)w^*(\xi) d\xi + \int_0^h d_1(\xi)w^*(\xi + h) d\xi = \frac{c}{16}, \quad (212)$$

$$Q_4 = \int_0^h d_4(\xi)w^*(\xi) d\xi + \int_0^h d_2(\xi)w^*(\xi + h) d\xi = \frac{c}{960}, \quad (213)$$

$$Q_5 = \int_0^h d_3(\xi)w^*(\xi + h) d\xi + \int_0^h d_1(\xi)w^*(\xi + 2h) d\xi = \frac{c}{8}, \quad (214)$$

$$Q_6 = \int_0^h d_4(\xi)w^*(\xi + h) d\xi + \int_0^h d_2(\xi)w^*(\xi + 2h) d\xi = \frac{c}{960}, \quad (215)$$

$$Q_7 = \int_0^h d_3(\xi)w^*(\xi + 2h) d\xi + \int_0^h d_1(\xi)w^*(\xi + 3h) d\xi = \frac{3c}{16}, \quad (216)$$

$$Q_8 = \int_0^h d_4(\xi)w^*(\xi + 2h) d\xi + \int_0^h d_2(\xi)w^*(\xi + 3h) d\xi = \frac{c}{960}, \quad (217)$$

$$Q_9 = \int_0^h d_3(\xi)w^*(\xi + 3h) d\xi = \frac{37c}{320}, \quad (218)$$

$$Q_{10} = \int_0^h d_4(\xi)w^*(\xi + 3h) d\xi = -\frac{3c}{640}. \quad (219)$$

Finally, the boundary conditions (4) written here equivalently as

$$\eta(0) = 0, \quad m(0) = 0, \quad \eta(1) = 0, \quad \theta(1) = 0 \quad (220)$$

are also taken into consideration in the final system of  $p$  ( $p = 2n + 2$ ) linear algebraic equations

$$\mathbf{KU} = \mathbf{F} \quad \text{with} \quad \mathbf{F} = \mathbf{Q} + \mathbf{R} \quad (221)$$

formed. In these equations,  $\mathbf{K}$  is the computed  $p \times p$  assembled stiffness matrix (for  $n = 4$  displayed in Eq. (205)),  $\mathbf{U}$  is the (mainly) unknown generalized nodal displacement vector (with  $p$  elements and including both the deflections  $\eta_l$  and the rotations  $\theta_l$  at the  $n + 1$  finite element nodes on the beam) here with  $u_1 = 0$ ,  $u_{p-1} = 0$  and  $u_p = 0$ ,  $\mathbf{F}$  is the (total) generalized nodal force vector,  $\mathbf{Q}$  is the computed nodal force vector due just to  $w^*(\xi)$  (also with  $p$  elements and for  $n = 4$  finite elements given in Eqs. (210)–(219)) and, finally,  $\mathbf{R}$  is the (mainly) known external nodal force vector (again with  $p$  elements) with just its elements  $R_1$ ,  $R_{p-1}$  and  $R_p$  unknown and the remaining elements equal to zero because no concentrated nodal force or moment loading is present at the nodes on the beam.

The system of linear algebraic equations (221) (in this section based on the method of finite elements) and (220) has been used here instead of the system of linear algebraic equations (5) and (6) in the previous section (based on the method of finite differences) for the derivation of the approximate ranges (intervals) of the unknown functions  $\eta(\xi)$  and  $\theta(\xi)$  on the beam continuously using REDUCE and REDLOG and following exactly the same approach as in the previous section.

The related simple set of commands prepared again with *Notepad* and used in REDUCE for quantifier elimination continuously using the REDLOG computer logic package in the present beam problem is the following set of commands (here with  $n = 4$  finite elements on the beam):

---

```
load_package "redlog"; rlset ofsf$
operator q,u,y,th,r,eqn;
n:=4$ h:=1/n;
ke:=mat((12/h^3,6/h^2,-12/h^3,6/h^2),
(6/h^2,4/h,-6/h^2,2/h),
(-12/h^3,-6/h^2,12/h^3,-6/h^2),
(6/h^2,2/h,-6/h^2,4/h));
procedure d1(x); 1-3x^2/h^2+2x^3/h^3$
procedure d2(x); x-2x^2/h+x^3/h^2$
procedure d3(x); 3x^2/h^2-2*x^3/h^3$
procedure d4(x); -x^2/h+x^3/h^2$
procedure w(x); c*x$
p:=2*n+2;
matrix ke1(p,p), ke2(p,p), ke3(p,p), ke4(p,p), un(p,1), qf(p,1)$
for i:=1:4 do «for j:=1:4 do ke1(i+0,j+0):=ke(i,j)»$
for i:=1:4 do «for j:=1:4 do ke2(i+2,j+2):=ke(i,j)»$
for i:=1:4 do «for j:=1:4 do ke3(i+4,j+4):=ke(i,j)»$
for i:=1:4 do «for j:=1:4 do ke4(i+6,j+6):=ke(i,j)»$
k0:=ke1+ke2+ke3+ke4$ k0;
q(1):=int(d1(x)*w(x+0*h),x,0,h);
q(2):=int(d2(x)*w(x+0*h),x,0,h);
q(3):=int(d3(x)*w(x+0*h),x,0,h)+int(d1(x)*w(x+1*h),x,0,h);
q(4):=int(d4(x)*w(x+0*h),x,0,h)+int(d2(x)*w(x+1*h),x,0,h);
q(5):=int(d3(x)*w(x+1*h),x,0,h)+int(d1(x)*w(x+2*h),x,0,h);
q(6):=int(d4(x)*w(x+1*h),x,0,h)+int(d2(x)*w(x+2*h),x,0,h);
q(7):=int(d3(x)*w(x+2*h),x,0,h)+int(d1(x)*w(x+3*h),x,0,h);
q(8):=int(d4(x)*w(x+2*h),x,0,h)+int(d2(x)*w(x+3*h),x,0,h);
q(9):=int(d3(x)*w(x+3*h),x,0,h);
q(10):=int(d4(x)*w(x+3*h),x,0,h);
qt:=q(1)+q(3)+q(5)+q(7)+q(9);
mt:=q(2)+q(4)+q(6)+q(8)+q(10);
```

```

for i:=2:8 do r(i):=0$
for i:=1:p do eqn(i):=«for j:=1:p sum k0(i,j)*u(j)»-q(i)-r(i);
eqs:=for i:=1:p mkand eqn(i)=0;
for i:=0:p do r(i):=mkid(r,i);
for i:=0:p do u(i):=mkid(u,i);
var:={u1,u2,u3,u4,u5,u6,u7,u8,r1,r9,r10};
y1:=u3$ y2:=u5$ y3:=u7$ th0:=u2$ th1:=u4$ th2:=u6$ th3:=u8$
u1:=0; u9:=0; u10:=0;
ycon:=d1<=0 and d1<=y1<=d2 and d1<=y2<=d2 and d1<=y3<=d2 and d1<=d2$
thcon:=ph1<=0 and ph1<=th0<=ph2 and ph1<=th1<=ph2 and ph1<=th2<=ph2
and ph1<=th3<=ph2 and ph1<=ph2$
var1:={c,u2,u3,u4,u5,u6,u7,u8,r1,r9,r10};
on time; ansy:=rlqe ex(var1, eqs and 1<=c<=2 and ycon);
ansth:=rlqe ex(var1, eqs and 1<=c<=2 and thcon);
ansall:=rlqe ex(var1, eqs and 1<=c<=2 and ycon and thcon); off time;
soly1:=solve(part(ansy,2,1),d1); on rounded; soly1; off rounded;
soly2:=solve(part(ansy,1,1),d2); on rounded; soly2; off rounded;
solth1:=solve(part(ansth,2,1),ph1); on rounded; solth1; off rounded;
solth2:=solve(part(ansth,1,1),ph2); on rounded; solth2; off rounded;

```

Here we denote by the symbol  $\mathcal{E}_2$  (again eqs in REDUCE) the  $p = 2n + 2$  (here we have  $n = 4$  and, therefore,  $p = 10$ ) linear finite element equations (221) of the present beam problem. The list of variables  $\mathcal{V}_1$  in Eqs. (8), the assumptions  $\mathcal{A}_1$  in Eq. (9), i.e.  $1 \leq c \leq 2$ , and the conditions  $\mathcal{C}_\eta$  in Eq. (11) remain the same. Then we have the purely existential quantified formula (12), but here simply with the symbol  $\mathcal{E}_2$  instead of the symbol  $\mathcal{E}_1$  with respect to the equations used.

In quite a similar manner, we can have the conditions  $\mathcal{C}_\theta$  related to the rotation (approximately the slope)  $\theta$  of the beam, of course, here again related to the finite element method as well as the nodes  $\eta_l = l/n$  on the beam. These conditions  $\mathcal{C}_\theta$  have again the form (13) and the related purely existential quantified formula has the form (14), but here again simply with the symbol  $\mathcal{E}_2$  instead of the symbol  $\mathcal{E}_1$  with respect to the equations used.

Obviously, we can also assume the simultaneous validity of all the conditions  $\mathcal{C}_\eta$  and  $\mathcal{C}_\theta$ . Then we have the sufficiently more difficult (from the computational point of view) quantified formula

$$\exists \mathcal{V}_1 \text{ such that Eqs. } \mathcal{E}_2 \text{ hold true under the assumptions } \mathcal{A}_1 \text{ and the conditions } \mathcal{C}_\eta \text{ and } \mathcal{C}_\theta. \quad (222)$$

The approximate results for the intervals (the ranges) of the two quantities  $\eta$  (dimensionless deflection) and  $\theta$  (rotation) were obtained by using the present method of finite elements and the quantifier-elimination-based approach with REDLOG with  $n = 4$  finite elements on the beam. The approximate interval (the approximate range) of the dimensionless deflection  $\eta$  of the beam is

$$\eta \in \left[ 0, \frac{3}{1280} \right] \approx [0, 0.002343750000000]. \quad (223)$$

An analogous interval was also obtained for the rotation  $\theta$  of the present beam again with  $n = 4$  finite elements. This approximate interval (the approximate range) of the rotation  $\theta$  of the beam is

$$\theta \in \left[ -\frac{203}{30720}, \frac{1}{120} \right] \approx [-0.00660807291667, 0.008333333333333]. \quad (224)$$

Taking into account the theoretical (exact) range (56) of the rotation  $\theta$ , we observe that the right endpoint of the above approximate range (224) is exact exactly as is the case with the trivial (here equal to zero) left endpoint of the range of the dimensionless deflection  $\eta$  of the beam in Eq. (223). Evidently, completely analogously, we can work with a larger number  $n$  ( $n > 4$ ) of finite elements.

## 5. Conclusions–discussion

From the above results it is concluded that the computational method of quantifier elimination can be successfully applied to the determination of intervals (ranges) of quantities of mechanical importance on the basis of the popular methods of finite differences and of finite elements. These intervals appear because of the presence of an uncertain overall loading parameter in the mechanical problem under consideration here a classical beam. In this problem, the computed intervals (ranges) concerned (i) the dimensionless deflection of the beam, (ii) the rotation of the beam and (iii) the dimensionless bending moment of the beam (this moment only with the finite difference method). Clearly, because of the use of the approximate methods of finite differences and of finite elements, the derived intervals (ranges) are also approximate, but, naturally, they are seen to converge to the expected theoretical intervals (ranges) as the number of intervals (elements) on the beam increases.

It is well known that the method of quantifier elimination generally concerns a small total number of variables (both free variables and quantified variables) in the quantified formulae; see, e.g., the classical paper by Davenport and Heintz [6], where it is proved that real quantifier elimination is doubly exponential. Here it has become possible to use up to 3072 intervals on the beam, i.e. a total number of 3076 quantified variables in the quantified formulae (for all three quantities of interest), in the case of a purely existentially quantified formula, i.e. with an existential quantification of the uncertain overall loading parameter  $c$  of the beam ( $\exists c \in [1, 2]$ ). This number was reduced to 512 intervals, i.e. 516 quantified variables in the quantified formulae (for all three quantities of interest), in the case of a mixed universally–existentially quantified formula, i.e. with a universal quantification of the same overall loading parameter  $c$  of the beam ( $\forall c \in [1, 2]$ ). This reduction seems to be due to the presence of both quantifiers  $\forall$  and  $\exists$  in the second and less favourable case. Nevertheless, even the number of 516 total variables is really an extremely large number in quantifier elimination.

The success of the present approach is simply due to the fact that here we used only an uncertain loading parameter  $c$  (belonging to the interval  $[1, 2]$ ), i.e. a parameter only in the right-hand sides of the system of parametric interval linear algebraic equations. This situation had as a consequence the linearity of this system not only with respect to the unknowns, but also with respect to the parameter  $c$ , i.e. with respect to all variables. In this case, we abandoned the classical method of using the general-purpose CAD (cylindrical algebraic decomposition) algorithm of Collins [3, 4] usually used in quantifier elimination for real variables in favour of the method of virtual substitution of Weispfenning [7–10], the applicability of which is generally restricted to the linear and the quadratic cases, here to the very favourable linear case. For this reason here we selected to use the REDLOG computer logic package [16] of the REDUCE computer algebra system [11], which mainly uses the virtual substitution method for quantifier elimination (although CAD can also be used) instead of *Mathematica* [5], which mainly uses CAD for the same task although virtual substitution is also automatically used in simple cases such as the present linear case. Of course, it is probably worthwhile considering the use of *Mathematica* for the present computational tasks after instructing it not to use CAD, but to use virtual substitution instead, through the use of the related commands.

It is understood that the present results concern an uncertain (interval) parameter  $c$  (the overall loading parameter of the beam) belonging to an interval uniformly in the right-hand sides of the linear algebraic equations. Nevertheless, it is also completely possible to have an uncertain parameter in some of these right-hand sides or even more than one uncertain (interval) parameter in the same right-hand sides provided, of course, that no products of these parameters (and, obviously, more generally, no products of the quantified variables) appear in the whole parametric interval system of linear algebraic equations. Finally, evidently, although the present results concern a simple beam problem, nevertheless, they are also applicable to more difficult applied and computational mechanics problems concerning the finite difference method or the finite element method or any other method in computational mechanics which similarly leads to systems of linear algebraic equations.

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