Department of Computer Engineering and Informatics

University of Patras

Fair division of indivisible goods

Graduate program "Computer Science and Technology"

MSc Thesis

Stavros D. Ioannidis

Advisor: Ioannis Caragiannis, Professor

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Thesis committee:

Ioannis Caragiannis (Advisor)

Professor, University of Patras

Stavros Cosmadakis

Professor, University of Patras

Sotirios Nikoletseas

Professor, University of Patras

Dedicated to my beloved parents

Despoina and Dimitris

Preface

It is widely notable the last 20 years at least, that computer science as a field has invaded into many other scientific fields. It suffices only to take a look at recent and elder computer science study programs to identify that computer science is not only about the computer nowadays but also about economics, business, social sciences and more. But what is the cause of this diversity?

Of course the birth of a new scientific field, especially one at the edge of mathematics like computer science, always inflates a domino of new ideas and concepts until we point out the true questions and barriers of the field. Such a barrier is the 'P vs NP' problem whose solution seems to bother scientists for a long period of time. So even though we have discovered the true barriers of computation and the computer, how come computer science be so loudly around. It seems like something arose something that attracted everyone's attention from the beginning a true milestone that changed the direction of the field. That milestone was the Internet. If we have to convince someone about the importance of the Internet, just let him consider that many key scientists of the field speak about the Internet as a true revolution and moreover with the Internet computation has invaded into social sciences.

So how much does the Internet affect recent research? Internet makes us all participants of a new bigger and more demanding society. This society needs to be studied and researched in order to function optimally. Here is an example: Let us say that a matter of global interest arises, a matter in which everyone should have an opinion that must be taken under consideration. Using the global network now we can make this happen. We can construct voting algorithms to help us hold an international voting. So we need to investigate ways to lead us to fast voting algorithms with good performance guarantees. This is just a particular example. Consider online markets, online auctions, online negotiations about items and goods, resource allocations between people, companies, governments globally. Generally think about a future where social problems and matters would be solved by intelligent and fast AI algorithms.

As we mentioned above, computer science has invaded many other scientific fields. A first attempt was on economics and especially game theory where the combination of concepts from both directions gave birth to a new field called algorithmic game theory. Another trend that arose the last years is that of mixing algorithmic ideas and concepts with social choice issues. Binding concepts from social choice, welfare economics, decision theory with AI, complexity-analytic algorithms design, mechanism design, multi-agent system negotiations, composes computational social choice. Some of the major problems being researched in this field are: Preference aggregation, voting theory, resource allocation, fair division, coalition formation, judgement aggregation, belief merging and ranking systems and more.

In this thesis we study fair division problems. Fair division problems are categorized generally as resource allocation problems where we have to divide items to players or agents satisfying specific fairness notions. The task depends on many parameters like the nature of the items we allocate to agents which may be either divisible or indivisible, the functions we use to model the agents preferences, whether a central algorithm runs for the problem or the agents themselves converge to a solution namely whether the problem is centralized or distributed. Here we study only the case where the items are indivisible and hence, they must be allocated as a whole to some agent.

This thesis is organized in three Chapters. Chapter 1 is about the classic setup of a fair division problem of indivisible goods. We present the classic model of the problem, we give definitions about the functions that model agents' preferences, we discuss ways to measure system efficiency inheriting notions from welfare economics, we present some of the most commonly used fairness notions like proportionality and envy-freeness and finally, we present some recent theorems and results on these notions.

Chapter 2 deals with distributed fair division. Here the agents negotiate rational deals on exchanging goods that benefit them. We present the model of distributed fair division and mark the key differences from Chapter 1. We give a summary of results from some key papers and close with some fairness results and negotiations in general.

The final Chapter, Chapter 3 deals with results that we discovered while working on this thesis concerning fair division problems using subsidy. We study the minimum subsidy required for an allocation to be envy-freeable and conclude with an approximation algorithm and a hardness result.

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Contents

Chapter 1

Centralized fair division

1.1 Abstract

In this chapter we are about to discuss and present centralized fair allocation of indivisible goods. These problems lie in the field of fair division problems which is a part of computational social choice research. In these problems, there is a set of goods and a set of agents, to whom the goods must be allocated to. Each agent has some valuation for every good, which depicts how much the agent wants the good. Based on these valuations we will try to find allocations that meet certain criteria.

We would like our allocation to be fair. The notion of fairness is defined and is proved to be very demanding in our framework. Apart from fairness, we would like our allocations to have high efficiency guarantees that means we would like our allocations to be economically good. Both economic efficiency and fairness are defined in this chapter. The term "centralized" means that there is a central authority to allocate the goods to the agents. This central authority would be an algorithm that takes as input the number of goods and agents and the valuation the agents have over the goods and computes an allocation based on some fairness criteria.

This chapter is organised as follows. First we present the model which we work on and the kind of valuation functions the agents use to express their preferences over the set of goods. Next we define some economic efficiency measures we are about to use in the rest of our survey. There is a section in which we define some basic fairness criteria and fairness notions that are commonly used in literature. A special section of examples follows to illustrate theory and notions already

defined. Finally we conclude by presenting some recent results on the field based on papers we read throughout the course of this thesis. We begin the chapter right away by presenting the model of the classical instance of a fair division of indivisible goods problem.

1.2 The model

Let M = $\{g_1, g_2, g_3, \dots, g_k\}$, $k \geq 0$ be a finite set. We call every element $g_i \in M$ a good or item. We call M the set of goods. We call every subset B of M a bundle of goods. Note that every good g_i is to be considered as an indivisible good meaning that it must be allocated to an agent as it is and not divided.

Let N = $\{a_1, a_2, a_3, \dots, a_n\}$, $n \geq 0$. We call every element $a_i \in N$ an agent and the set N the set of agents. Some times we will refer to an agent a_i as i for simplicity and instead of $a_i \in N$ we will write $i \in N$.

We denote by $\Pi_n(M)$ the set of all ordered partitions consisting of n subsets of M. Any element of $\Pi_n(M)$ models an allocation of goods from M to N. We will denote an allocation of goods by A where $A = (A_1, A_2, A_3, \dots, A_n)$ with $\bigcup_{i=1}^{n} A_i = M$ and $\bigcap_{i=1}^{n} A_i = \emptyset$. Any allocation A consists of a set of n bundles, A_i, where A_i denotes the bundle of goods allocated to agent i in allocation A. We point out from the above relations that no two agents can share a same good and every good has to be allocated to some agent.

Every agent has some preference for the goods in set M. We will use these preferences to design algorithms that solve fair division problems. To model these preferences we use functions called valuation functions. A valuation function assigns each bundle of goods to the non-negative reals, $\mathbb{R}_{\geq 0}$ and might be different for each agent. We denote by v_i the valuation function of agent i. There are many kinds of valuation functions on which we will elaborate later on.

The above model is the general model of indivisible good allocation problems. Given an instance of this model, our purpose will be to figure out algorithms that distribute the goods to the agents fairly.

1.3 Valuation Functions

As we mentioned in the description of the model, we use valuation functions to model the agent preferences over the bundles, hence these functions are defined over 2^M. We refer to v_i as the valuation function of agent i. Valuation functions are very important in our study, since it will often be the case that one result might hold for a specific kind of valuation functions only. Some kinds of valuation functions are mentioned below.

Definition. *A valuation function v_i: $2^M \mapsto \mathbb{R}_{\geq 0}$ is called submodular if for every pair of bundles $S, T \subseteq M$,*

$$v_i(S \cup T) \leq v_i(S) + v_i(T) - v_i(S \cap T).$$

Respectively, a valuation function is called supermodular if

$$v_i(S \cup T) \geq v_i(S) + v_i(T) - v_i(S \cap T).$$

Definition. *A valuation function, v_i: $2^M \mapsto \mathbb{R}_{\geq 0}$ is called subadditive if for every disjoint pair of bundles $S, T \subseteq M$,*

$$v_i(S \cup T) \leq v_i(S) + v_i(T).$$

Respectively, a valuation function is called superadditive if

$$v_i(S \cup T) \geq v_i(S) + v_i(T).$$

Definition. *A valuation function v_i: $2^M \mapsto \mathbb{R}_{\geq 0}$ is additive if $\forall\ S \subseteq M$,*

$$v_i(S) = \sum_{g \in S} v_i(g).$$

Definition. *A valuation function v_i: $2^M \mapsto \mathbb{R}_{\geq 0}$ is called single-minded if there exists a bundle $S \subseteq M$, and $c > 0$ such that*

$$v_i(B) = \begin{cases} c & \text{if } S \subseteq B \\ 0 & \text{otherwise} \end{cases}$$

meaning we are only interested in bundle S.

Definition. *A valuation v_i is called binary if $v_i(g) \in \{0, 1\}$, $\forall g \in M$.*

For convenience, we will denote the valuation of single goods as $v_i(g)$ instead of $v_i(\{g\})$ and agent a_i by i. Note also that in additive valuation functions it suffices only to determine the valuation on each good separately instead of every bundle. Lastly, we consider the valuation for the empty set to be zero for every agent, $v_i(\emptyset) = 0$, $\forall i \in N$. The last assumption alongside with non-negativity characterizes these valuations as normalised.

There are concepts and variations of the classical problem, where the agents do not actually gain the complete valuation from the goods they receive. By that we mean that the agents, perhaps have to detach some amount of valuation from the received bundle, and they do so by paying some amount of money. To model the complete valuation each agent actually gains after money subtraction we use utility functions.

We denote by u_i the utility function of agent i. In our study the utility function is defined by the valuation function. In chapter 2 we will refer to utility as valuation minus money, namely we will denote $u_i(B) = v_i(B) - p_i$, to be the utility of agent i when he receives bundle B and has to pay p_i amount of money.

For purposes of completeness, we presented briefly the notion of utility functions. As far as this chapter is concerned the utility of an agent will be the full valuation he receives by the bundle that is being allocated to him. So $u_i(B) = v_i(B)$, $\forall B \subseteq M$,$\forall\ i \in N$ throughout chapter 1.

1.4 Efficiency

As we mentioned earlier many concepts from economics are inherited in computational social choice. Economic efficiency is such a concept. The ideal outcome of any algorithm and mechanism would be an allocation with high efficiency guarantees.

By efficiency we mean how much satisfied the system (agents) is with respect to a specific efficiency measure. Consider that in any allocation problem, our a-priori knowledge is the valuation function of agents and after each allocation we have information for the utility of each one. How can we use this information to compute highly efficient allocations for the agents? In this section we will present the efficiency measures used in fair division problems, many of them are being studied in welfare economics. We will also present definitions of popular social welfare functions and comment on them. Of course the only available tools we already have in building efficiency measures are the utility functions of the agents.

1.4.1 Pareto efficiency

Pareto efficiency is a benchmark notion used in welfare economics to study the efficiency of a system like the classical resource allocation instance. Pareto efficiency is named after economist Vilfredo Pareto. First let us give the definition of a utility vector we are about to use afterwards.

Definition (**Utility vector**). *Given an allocation* $A = (A_1, A_2, A_3, \ldots, A_n)$ *from a set of goods* M *over a set* N *of* n *agents we call* $u = \langle u_1, u_2, u_3, \ldots, u_n \rangle$ *the utility vector of allocation* A *where* $u_i = u_i(A_i)$ *is the utility of agent* i *when he receives bundle* A_i*. Every allocation* A *has an induced utility vector.*

Before defining Pareto efficiency we give the definition of Pareto dominance, which leads naturally to Pareto efficiency.

Definition (**Pareto dominance**). *Given 2 allocations* A*,* A' *we say that* A *is Pareto dominated by* A' *if and only if* $u_i(A_i) \leq u_i(A'_i)$*,* $\forall\ i \in N$ *and there is at least one agent say* k *for which the inequality is strict.*

Definition (**Pareto efficient**). *An allocation* A *is Pareto efficient or Pareto optimal (PO) if and only if there is no other allocation* A' *such that* A *is Pareto dominated by* A'*.*

Formally speaking, Pareto dominance is a binary relation, call it PD, over the positive space $\mathbb{R}^n$. Assuming utility vectors a and b corresponding to allocations A and B respectively, $(a, b) \in PD$ if and only if allocation A is Pareto dominated by allocation B. It is easy to prove that PD is a strict partial order over the positive n-dimension vectors (meeting irreflexivity, transitivity, asymmetry).

So, PD orders the set of allocations (via their utility vectors) partially. It is obvious that there might be allocations that cannot be compared under this order and also this order has maximal elements. These maximal elements are the PO allocations we want. Namely what PO depicts is that no reallocation of goods can make each agent at least as happy as he already is.

Although PO seems to be a desired efficiency measure, it is rather weak. Consider an allocation where one agent takes everything. By definition, this is already a Pareto optimal allocation (assuming non-zero valuations). So not only there might exist many PO allocations, there are PO

allocations that are not fair at all. This can be pointed out even though we have not formally discussed about fairness yet.

1.4.2 Social welfare functions

Of course, in Pareto optimality we take information only from each agent's utility separately, while we can find functions of these utilities to give us more information about the system. Such functions are called collective utility functions (CUF) or as we will refer to them social welfare functions (SW).

Definition. *A social welfare function SW:* $\mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ *denoted as* SW*, is a function mapping a utility vector to the non-negative reals.*

Every SW induces an ordering called Social Welfare ordering, SWO. Given a social welfare function and two utility vectors u, v we have $u \leq v$ if and only if $SW(u) \leq SW(v)$. Below we give some important social welfare functions.

Definition (Utilitarian social welfare). *The utilitarian social welfare function,* SW_{util} *maps a utility vector to the sum of its components:*

$$SW_{util} = \sum_{i \in N} u_i.$$

Definition (Egalitarian social welfare). *The egalitarian social welfare,* SW_{egal} *maps a utility vector to its minimum valued component:*

$$SW_{egal} = \min\{u_i | i \in N\}.$$

Definition (Elitist social welfare). *The elitist* SW*,* SW_{elit} *maps a utility vector to its maximum valued component:*

$$SW_{elit} = \max\{u_i | i \in N\}.$$

Both egalitarian and elitist social welfare functions are special cases of a wider family called k-ranked dictators, where a utility vector is mapped to its k-th valued component.

A social welfare function of great significance, combining efficiency and fairness is the Nash social welfare (known as Nash product).

Definition (Nash social welfare). *The Nash social welfare function, SW_{nash} maps a utility vector to the product of its components:*

$$SW_{nash} = \prod_{i \in N} u_i.$$

1.5 Fairness

1.5.1 Proportionality and Envy-freeness

In fair division problems the objective is to find allocations that have high efficiency guarantees. Usually we want to maximize some social welfare function of the above group. Apart from efficiency though, fairness is another objective we try to sustain. The combination of efficiency and fairness would be ideal. In the same way we defined efficiency using the valuation functions of the agents, we will define fairness as well. What classifies an allocation as fair though?

In any allocation each agent desires to posses the highest-valued bundle, according to his preferences. That means he does not care about the other bundles that have less value, than the one he possesses, but only for the ones he values higher and are owned by other agents. This informally describes the notion of envy-freeness, a notion widely used to characterize fair allocations.

Something else we can think on fairness is the so called proportionality notion. Assuming three kids have to divide a chocolate between them in a fair way. Obviously one might think dividing the chocolate in three equal pieces is the best way. That way every kid gets at least $\frac{1}{3}$ of the whole piece. What is less obvious though is that not every kid has the same preference for the chocolate thus not every $\frac{1}{3}$ is the same. This statement needs more attention. Below we are about to give the definitions of some fairness criteria used in literature.

Definition (Proportionality). *An allocation A is proportional if and only if $v_i(A_i) \geq \frac{v_i(M)}{n}$, $\forall i \in N$.*

Proportionality states what we briefly mentioned above, that if an agent receives at least valuation equal to $1/n$ of his valuation for the whole set of goods he is satisfied. Proportionality may be a strong or weak fairness notion. It depends on the kind of valuation functions used. For additive valuations, proportionality is a desired notion. Another important fairness criterion is envy-freeness.

Definition (Envy-freeness). *An allocation A is envy-free if and only if $v_i(A_i) \geq v_i(A_j)$, $\forall i, j \in N$.*

Envy-freeness seems to be an obvious fairness criterion, meaning that every agent values his bundle at least as much as every other agent's bundle. If in a specific allocation somebody wants somebody else's bundle more, then he envies him. We would like our allocations to be envy-free. We refer to envy-free allocations as EF.

Unfortunately, EF allocations do not always exist. Consider as an example an instance with two agents and one good. Assuming nonzero valuations for the good, no allocation is EF. Any algorithm would allocate the good to either one of the two agents. Since the other one has positive valuation for the good, the allocation cannot be EF.

1.5.2 Envy-freeness approximations

The question that arises is whether there are allocations and measures that approximate envy-freeness and whether these notions are useful in theory. Indeed, such approximate measures have been proposed in the literature. Some among them are $EF1$, MMS, $PMMS$, EFX which we are about to define next. We begin by $EF1$, a criterion that is guaranteed to always exist in every instance proposed by Budish in [6].

Definition (EF1, Envy-freeness up to one good). *An allocation A is called EF1 if and only if $\forall\ i, j \in N$, $\exists\ g \in A_j$ s.t $v_i(A_i) \geq v_i((A_j) \setminus \{g\})$.*

It is obvious that any EF allocation is by definition $EF1$ also. An $EF1$ allocation allows some agents to envy others, but by removing one specific good, enviousness removes also. The relation in the definition is equivalent to the relation $v_i(A_i) \geq v_i(A_j) - \max\{v_i(g) | g \in A_j\}$.

Apart from $EF1$, envy-freeness up to k goods (EFk) can be defined in the same analogy. We know that we can always find $EF1$ allocations which we will present in the next section. Another stronger fairness notion than $EF1$ is that of EFX proposed by Caragiannis et al in [9].

Definition (EFX, Envy-freeness up to any good). *An allocation A is EFX if and only if $\forall i, j \in N$ and $\forall\ g \in A_j$ for which $v_i(g) > 0$: $v_i(A_i) \geq v_i(A_j \setminus \{g\})$.*

Note that we take under consideration only non zero valued goods. EFX states that an agent is envious of another agent, say k, but if we remove any good from k's bundle we remove enviousness. The inequality in the definition of EFX is equivalent to the next relation, $v_i(A_i) \geq v_i(A_j) - \min\{v_i(g) | g \in A_j\}$.

EFX allocations are more desired than $EF1$. Unfortunately not many things have been shown for EFX allocations (at least until the time this thesis is written), we do not know yet if an EFX allocation always exists in every instance.

Another EF relaxation is $\alpha \cdot MMS$ proposed in [6] (maximin share guarantee).

Definition ($\alpha \cdot MMS$**:** α**-maximin share guarantee**)**.** *The maximin share guarantee of agent* i*,* MMS_i*, is defined as*

$$MMS_i = \max_{A \in \Pi_n(M)} \min_{k \in [n]} \{v_i(A_k)\}$$

where $[n] = \{1, 2, \ldots, n\}$*. We say that an allocation* A *is* $\alpha \cdot MMS$ *if and only if* $v_i(A_i) \geq \alpha \cdot MMS_i . \forall i \in N$

The maximin share guarantee actually is given by forming a set consisting of the lowest valuation an agent gets in every possible allocation and picking the maximum of this set. For our presentation we assume that $\alpha = 1$.

This is a classical cut-and-choose idea, where one agent forms an allocation and picks his bundle last. In that way the agent can guarantee his maximin share since he might end up choosing his least valued bundle, which he formed so as to be as high as possible. Cut-and-choose is an idea from fair division of divisible goods (e.g. see [5])

An even stronger fairness notion exists called pairwise maximin share guarantee $PMMS$ defined by Caragiannis et al in [9].

Definition (**Pairwise maximin share guarantee**)**.** *We say that an allocation* A *is a pairwise maximin share if and only if* $\forall\ i, j \in N$*,* $v_i(A_i) \geq \max_{B \in \Pi_2(A_i \cup A_j)} \min\{v_i(B_1), v_i(B_2)\}$

Note that pairwise-MMS is similar to MMS (for instances with two agents, they are actually identical) but instead of agent i partitioning the set of all goods into n bundles he partitions the combined bundle of himself and another agent into two bundles and receives the one he values less.

Again this is a cut-and-choose, in the manner that the agent will formulate the most highly least valued bundle in case he has to actually pick the least valued bundle.

So far, we have mentioned some fairness notions and approximations that are used in the literature. Many more exist like $MEF1$ defined by Caragiannis et al in [9]. For additive valuations $MEF1$ is $EF1$.

Definition (**Marginal Envy Freeness up to One Good**). *An allocation A satisfies $MEF1$ if and only if*

$$\forall i, j \in N, \exists g \in A_j, v_i(A_i) \geq v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i).$$

As we have already mentioned, many concepts and proofs depend on the valuation function. It will often be the case that a theorem holds for additive but not for other types of valuations. $MEF1$ is defined in a way so as to help us establish a theorem for non-additive valuations.

Similarly to the maximin share guarantee, the min-max fair share is defined as follows.

Definition. *The min-max fair share of agent i, denoted as u_i^{mFS}, is defined as:*

$$u_i^{mFS} = \min_{A \in \Pi_n(M)} \max_{k \in [n]} \{v_i(A_k)\}.$$

An allocation A satisfies the min-max fair share if and only if, $u_i(A_i) \geq u_i^{mFS}$, $\forall i \in M$.

One final EF relaxation we present here is the epistemic envy-freeness EEF defined by Aziz et al in [3].

Definition (**Epistemic Envy-freeness**). *An allocation A is EEF if and only if $\forall i \in N$ there exists an allocation say A^i s.t $A_i = A_i^i$ and $v_i(A_i) \geq v_i(A_j^i)$, $\forall j \in N$.*

Roughly speaking, an allocation A is EEF if for every agent i the goods that are not allocated to him can be re-allocated to the rest of the agents so as agent i does not envy anyone after the re-allocation.

1.6 Examples

In this section we will give some examples to illustrate the above definitions.

Example 1. Consider the following instance with three goods and two agents, $M = \{g_1, g_2, g_3\}$ and $N = \{1, 2\}$. We give the valuations of the agents for each good in the form of the table below.

Agents	g_1	g_2	g_3
1	1	1	2
2	1	1	1

In the table we have the set of agents, the set of goods and the valuations for them. Subsets $g_1, \{g_1, g_3\}, \emptyset, \{g_1, g_2, g_3\}$ are all possible bundles of set M. Partitions $(g_1, \{g_2, g_3\})$, $(\{g_1, g_2\}, g_3)$, $(\emptyset, \{g_1, g_2, g_3\})$ are some feasible allocations where in the first agent 1 receives good g_1 and agent 2 receives bundle $\{g_2, g_3\}$, while in the last agent 1 receives nothing and agent 2 receives everything. We use the letter A to denote an allocation, for example $A = (g_2, \{g_1, g_3\})$ is an allocation where agent 1 receives good g_2 and agent 2 receives bundle $\{g_1, g_3\}$.

For agent 1 $v_1(g_1) = 1$ is his valuation for good g_1, $v_1(g_2) = 1$ his valuation for good g_2 and $v_1(g_3) = 2$ his valuation for good g_3. For agent 2 $v_2(g_1) = 1$ is his valuation for good g_1, $v_2(g_2) = 1$ his valuation for good g_2 and $v_2(g_3) = 1$ his valuation for good g_3.

Since we assume additive valuations, we can determine valuation over all bundles. For example
$v_1(\{g_1, g_3\}) = v_1(g_1) + v_1(g_3) = 1 + 2 = 3$
$v_2(\{g_1, g_2, g_3\}) = v_2(g_1) + v_2(g_2) + v_2(g_3) = 1 + 1 + 1 = 3.$

Now consider allocation $A = (\{g_1, g_2\}, g_3)$ and allocation $A' = (g_3, \{g_1, g_2\})$. The induced utility vectors for these allocations are $u_A = \langle v_1(\{g_1, g_2\}), v_2(g_3)\rangle = \langle 2, 1\rangle$ and $u_{A'} = \langle v_1(g_3), v_2(\{g_1, g_2\})\rangle$ $= \langle 2, 2\rangle$.
Allocation A is Pareto dominated by allocation A'. A' is also Pareto optimal. Allocations $(\emptyset, \{g_1, g_2, g_3\})$ and $(\{g_1, g_3\}, g_2)$ are also Pareto optimal allocations.

Next consider allocation $A = (\{g_2, g_3\}, g_1)$. The utilitarian social welfare of allocation A is $SW_{util}(A) = v_1(\{g_2, g_3\}) + v_2(g_1) = 3 + 1 = 4$.
The egalitarian social welfare is $SW_{egal}(A) = \min\{v_1(\{g_2, g_3\}), v_2(g_1)\} = \min\{3, 1\} = 1$
The elitist social welfare of allocation A is $SW_{elit}(A) = \max\{v_1(\{g_2, g_3\}), v_2(g_1)\} = \max\{3, 1\} = 3$

The Nash social welfare of allocation A is $SW_{nash}(A) = v_1(\{g_2, g_3\}) \cdot v_2(g_1) = 3 \cdot 1 = 3$.

Consider allocation $A = (g_3, \{g_1, g_2\})$. A is proportional since for both agents holds $v_1(g_3) \geq \frac{v_1(M)}{2}$ and $v_2(\{g_1, g_2\}) \geq \frac{v_2(M)}{2}$ whereas $A' = (g_1, \{g_2, g_3\})$ is not because for agent 1 $v_1(g_1) = 1 < \frac{v_1(M)}{2}$.

Allocation A is EF since for agent 1 holds that $v_1(g_3) \geq v_1(\{g_1, g_2\})$ and for agent 2 also holds that $v_2(\{g_1, g_2\}) \geq v_2(g_3)$, remember that EF holds if for every agent i and agent j, $v_i(A_i) \geq v_i(A_j)$, where A_i is the bundle he receives and A_j the bundle the other agent receives. Allocation $A' = (g_2, \{g_1, g_3\})$ is not EF because for agent 1 we have $v_1(g_2) = 1 < v_1(\{g_1, g_3\}) = 3$. Notice that agent 2 is not envious of agent 1 since $v_2(\{g_1, g_3\}) = 2 > v_2(g_2) = 1$.

Since allocation A is EF it is also $EF1$ by definition. Allocation A' is not EF, it is $EF1$ though, firstly because agent 1 is envious and also because there exists a good which when removed makes agent 1 non-envious. If we remove good g_3 from agent 2, he is not envious anymore: $v_1(g_2) = 1 \geq v_1(\{g_1, g_3\} \setminus g_3) = 1$. Allocation $A'' = (\{g_1, g_2\}, g_3)$ is not EF since agent 2 is envious because $v_2(g_3) = 1 < v_2(\{g_1, g_2\}) = 1 + 1 = 2$. It easy to notice that A'' is $EF1$ by removing from agent 1 either good g_1 or good g_2.

Consider allocation $A = (\{g_2, g_3\}, g_1)$, obviously this allocation is not EF since agent 2 is envious of agent 1: $v_2(g_1) = 1 < v_2(\{g_2, g_3\}) = 1 + 1 = 2$. If we remove from agent 1 either good g_2 or g_3 he is not envious any more. So by removing any good from agent 1, agent 2 is not envious anymore which means that this allocation is not only $EF1$ but EFX too.

Computing the MMS_i of every agent demands more computations. Let's compute the MMS_i for the agents i of this instance. First we must form all allocations and then construct the set of valuations of the minimum valued bundle of each agent i in each one of these allocations. For agent 1 we get:

$(g_1, \{g_2, g_3\})$ where $\min\{v_1(g_1), v_1(\{g_2, g_3\})\} = \min\{1, 3\} = 1$

$(g_2, \{g_1, g_3\})$ where $\min\{v_1(g_2), v_1(\{g_1, g_3\})\} = \min\{1, 3\} = 1$

$(g_3, \{g_1, g_2\})$ where $\min\{v_1(g_3), v_1(\{g_1, g_2\})\} = \min\{2, 2\} = 2$

$(\{g_1, g_2\}, g_3)$ where $\min\{v_1(\{g_1, g_2\}), v_1(g_3)\} = \min\{2, 2\} = 2$

$(\{g_2, g_3\}, g_1)$ where $\min\{v_1(\{g_2, g_3\}), v_1(g_1)\} = \min\{3, 1\} = 1$

$(\{g_1, g_2, g_3\}, \emptyset)$ where $\min\{v_1(M), v_1(\emptyset)\} = \min\{4, 0\} = 0$

$(\emptyset, \{g_1, g_2, g_3\})$ where $\min\{v_1(\emptyset), v_1(M)\} = \min\{0, 4\} = 0$

For the MMS_1 we compute the max between $\{1, 2, 0\}$. So $MMS_1 = \max\{1, 2, 0\} = 2$. For agent 2 we have the same computations.

$(g_1, \{g_2, g_3\})$ where $\min\{v_2(g_1), v_2(\{g_2, g_3\})\} = \min\{1, 2\} = 1$

$(g_2, \{g_1, g_3\})$ where $\min\{v_2(g_2), v_2(\{g_1, g_3\})\} = \min\{1, 2\} = 1$

$(g_3, \{g_1, g_2\})$ where $\min\{v_2(g_3), v_2(\{g_1, g_2\})\} = \min\{1, 2\} = 1$

$(\{g_1, g_2\}, g_3)$ where $\min\{v_2(\{g_1, g_3\}), v_2(g_3)\} = \min\{2, 1\} = 1$

$(\{g_2, g_3\}, g_1)$ where $\min\{v_2(\{g_2, g_3\}), v_2(g_1)\} = \min\{2, 1\} = 1$

$(\{g_1, g_2, g_3\}, \emptyset)$ where $\min\{v_2(\{g_1, g_2, g_3\}), v_2(\emptyset)\} = \min\{3, 0\} = 0$

$(\emptyset, \{g_1, g_2, g_3\})$ where $\min\{v_2(\emptyset), v_2(\{g_1, g_2, g_3\})\} = \min\{0, 3\} = 0$

$MMS_2 = \max\{1, 0\} = 1$. After these computations we can now define the MMS allocations. For example $A = (\{g_1, g_2\}, g_3)$, $A' = (\{g_2, g_3\}, g_1)$, $A'' = (g_3, \{g_1, g_2\})$ are all MMS while $A''' = (g_1, \{g_2, g_3\})$ is not. Note that in this instance we only have two agents so $PMMS$ is the same as MMS by definition.

Example 2. We give an example of agents with different kind of valuation functions. Consider the following table.

Agents	g_1	g_2	g_3	$\{g_1, g_2\}$	$\{g_1, g_3\}$	$\{g_3, g_2\}$	$\{g_1, g_2, g_3\}$
1	5	5	0	10	5	5	10
2	4	4	4	8	8	8	16
3	4	4	4	8	8	8	10
4	0	1	1	1	1	2	2
5	0	0	0	8	0	0	8

All five agents of this instance follow different kind of valuation function. We are about to analyze them one by one. Agent 1 has additive valuation over the goods. Of course a full proof would be to examine each bundle separately, so as to conclude additivity. Agent 2 has supermodular valuation over the set of goods. Even though for bundle $\{g_1, g_2\}$ additivity holds, it does not hold for M. Check that for all three goods he has valuation 4 whereas for the whole bundle he has 16 instead of 12. Agent 3 in the same sense has submodular valuation function. Check again that his valuation of set M equals to 10 instead of 12. Again additivity holds for bundles $\{g_1, g_2\}$, $\{g_1, g_3\}$ and $\{g_3, g_2\}$. Agent 4 has binary valuation function over the goods. It only suffices to check whether his valuation for each single good is either 0 or 1. A useful remark that is not mentioned in the definition is that for a formed bundle the binary valuation function behaves as additive. Take a look at bundles $\{g_1, g_2\}$ and M, $v_4(\{g_1, g_2\}) = 1$ and $v_4(M) = 2$. Generally for binary valuations holds that $v_i(B) \leq |B|$, $\forall B \subseteq M$. Agent 5 has positive valuation for bundle $\{g_1, g_2\}$ only. Note also that $\{g_1, g_2\} \subseteq M$ and $v_5(M) = 2$, so his valuation is obviously single minded.

Example 3. Below we give the classic example of a non EF instance.

Agents	g_1
1	ε
2	ε

Before analyzing this example let's clarify that when using ε we will refer to a tiny but positive real number. So in the above instance both agents have valuation $\varepsilon > 0$ for the good. There are only two feasible allocations $A = (\{g_1\}, \emptyset)$ and $A' = (\emptyset, \{g_1\})$. In A $v_1(g_1) = \varepsilon$, $v_2(\emptyset) = 0$ and in A' $v_1(\emptyset) = 0$, $v_2(g_1) = \varepsilon$. It is straightforward that neither A nor A' are EF because in A $v_2(\emptyset) = 0 < \varepsilon = v_2(g_1)$ and in A'' $v_1(\emptyset) = 0 < \varepsilon = v_1(g_1)$. Of course both allocations are $EF1$ and EFX.

Example 4. Consider the following example with three agents and three goods.

Agents	g_1	g_2	g_3
1	2/3	1/3	0
2	1/3	2/3	0
3	1/3	1/3	1/3

Notice that every agent's valuation for the whole set of goods is 1. Notice also that there is a good g_3 for which agents 1 and 2 have zero valuation. Consider allocation $A = (g_3, g_1, g_2)$. A is

Pareto dominated by allocation $A' = (g_1, g_2, g_3)$. Also A' is PO because any reallocation of goods either gives a good to the agents or takes a good from them or swaps the goods between the agent. Any swap will give agents 1 and 2 less valuation since from 2/3 they receive at most 1/3. If we take a good from an agent and give it to another then one must receive no goods thus having zero valuation. If we give a good to an agent we must take away a good from another agent, which as we said previously doesn't work.

Notice also that the SW_{util} of allocation A' is the $2/3 + 2/3 + 1/3 = 5/3$ which is the highest utilitarian social welfare the system can get. Mark that every allocation that allocates good g_3 to agent 3 gives the maximum SW_{egal} the system can get. Obviously allocation A is not EF (check envy-free relation between agent 1 and 2 or between agent 1 and agent 3 or between agent 2 and 3). A is $EF1$ since every agent receives one good.

Next consider the allocation $A'' = (\{g_1, g_3\}, g_2, \emptyset)$. Agent 1 is not envious of anybody. Agent 2 is not envious of agent 1 and agent 3. Agent 3 is envious of both agents. This allocation is not even $EF1$. The reason is that the $EF1$ relation holds between agents 3 and 2 but it does not hold between agents 3 and 1. Notice that $v_3(\emptyset) = 0 < v_3(\{g_1, g_3\} \setminus g_1) = 1/3$ and $v_3(\emptyset) = 0 < v_3(\{g_1, g_3\} \setminus g_3) = 1/3$. There is no good to satisfy the $EF1$ relation.

To illustrate better the EFX notion consider the following instances:

Agents	g_1	g_2	g_3
1	2/3	1/3	0
3	1/3	1/3	1/3

Agents	g_1	g_2	g_3
1	2/3	1/3	ε
3	1/3	1/3	1/3

In the first instance agent 1 has zero valuation for good g_3 whereas in the second he has valuation $\varepsilon > 0$. In the first instance consider allocation $A = (g_2, \{g_1, g_3\})$. Agent 1 is envious of agent 3: $v_1(g_2) = 1/3 < v_1(\{g_1, g_3\}) = 2/3$. Agent 3 is not envious of agent 1: $v_3(\{g_1, g_3\}) = 2/3 > v_3(g_2) = 1/3$. Allocation A is also $EF1$ since $v_1(g_2) = 1/3 > v_1(\{g_1, g_3\} \setminus g_1) = 0$. Allocation A is also

EFX, since for agent 1 it suffices to remove item g_1 only. Remember in the EFX definition we take under consideration only the non-zero valued goods.

In the second instance consider the allocation $A' = (g_2, \{g_1, g_3\})$. A' is not EF since agent 1 is envious of agent 3 $v_1(g_2) = 1/3 < v_1(\{g_1, g_3\}) = 2/3 + \varepsilon$. Agent 3 is not envious of agent 1, $v_3(\{g_1, g_3\}) = 2/3 > v_3(g_2) = 1/3$. A' is $EF1$ for agent 1 since by removing good g_1 we satisfy the $EF1$ relation for agent 1. A' is not EFX and here is why: $v_1(g_2) = 1/3 > v_1(\{g_1, g_3\} \setminus g_1) = \varepsilon$. Now good g_3 is not zero valued for agent 1 and by the definition of EFX in order for A' to be EFX it must hold that $v_1(g_2) = 1/3 > v_1(\{g_1, g_3\} \setminus g_3) = 2/3$ which of course is a contradiction. So allocation A' is not EFX.

Example 5. Consider the following instance with four goods and three agents

Agents	g_1	g_2	g_3	g_4
1	10	6	6	1
2	10	6	6	1
3	1	6	6	10

Consider the allocation $A = (g_1, \{g_2, g_3\}, g_4)$. A is not EF since agent 1 is envious of agent 2: $v_1(g_1) = 10 < 12 = v_1(\{g_2, g_3\})$. Agents 2 and 3 are envy-free. A is also $EF1$ since $v_1(g_1) = 10 > v_1(\{g_2, g_3\} \setminus g_2) = 6$. Since agent 1 values goods g_2 and g_3 the same it is obvious that A is EFX too.

Remember that an allocation is characterized as EEF if and only if for every agent there exist a reallocation which preserves his bundle and shuffles the goods to the other agents so as the agent is not envious of them. Indeed check allocation $A' = (g_1, g_2, \{g_3, g_4\})$ where we take good g_3 from agent 2 and give it to agent 3. A' is a reallocation of A where agent 1 is envy-free hence allocation A is EEF.

Example 6. We end this section of examples presenting a combination of fairness and efficiency.

Agents	g_1	g_2	g_3
1	1/2	1/2	ε
2	2/5	2/5	1/5

We have defined some social welfare functions like egalitarian, elitist and utilitarian. By maximizing some social welfare functions we mean that we try to find allocations that have the maximum SW value over the others. In this specific example the allocation that maximizes the utilitarian social welfare is $A = (\{g_1, g_2\}, g_3)$ where $SW_{util} = v_1(\{g_1, g_2\}) + v_2(g_3) = 1 + 1/5 = 6/5$. Notice that A is not EF, moreover it is not even $EF1$ (check agent 2). So we can easily conclude that maximizing the utilitarian social welfare does not give fair allocations.

It easy to check that allocation $A' = (g_1, \{g_2, g_3\})$ maximizes the Nash social welfare since $SW_{nash} = v_1(g_2) \cdot v_2(\{g_1, g_3\}) = 1/2 \cdot 3/5 = 3/10$. We can easily check that A' is not EF since for agent 1 $v_1(g_2) = 1/2 < v_1(\{g_1, g_3\}) = 1/2 + \varepsilon$, it is $EF1$ though and Pareto optimal too.

1.7 Recent results in fair division

There are many results in fair division and as this is a very hot research topic many more are being proved every year. We choose to present only some of them, that we studied while working on this thesis, that also helped us understand better the general framework and has led us to new results. For some theorems the proofs have been omitted but there is a reference to the actual paper it originated. First we deal with the first fairness notions we presented which where proportionality and envy-freeness. Below we prove the relation that holds between them.

1.7.1 Proportionality and envy-freenees

Proposition 1. Every envy-free allocation is proportional assuming additive valuations.

Proof. Consider an arbitrary EF allocation say A. Since allocation A is EF it must be the case that $v_i(A_i) \geq v_i(A_j)\ \forall i, j \in N$. Adding over all bundles in the allocation we get $n \cdot v_i(A_i) \geq \sum_{A_k} v_i(A_k) = v_i(\bigcup_{i=1}^{n} A_i) \implies v_i(A_i) \geq \frac{1}{n} \cdot v_i(M)$. □

The opposite direction does not hold for additive valuations though and here is a counterexample. Consider the following instance:

Agents	g_1	g_2	g_3
1	(1)	2	0
2	1	(2)	0
3	1	1	(1)

Check allocation $\{g_1, g_2, g_3\}$. Obviously it is proportional but not envy-free for agent 1. We can prove though that for two agents proportionality provides envy-freeness under additive valuations.

Proposition 2. For two agents and additive valuations every proportional allocation is envy-free.

Proof. Assume two agents, an allocation A that is proportional and bundles A_1, A_2. For the agents it holds that $v_i(A_i) > \frac{v_i(M)}{2}$, $\forall i \in \{1, 2\}$. Assume A is not envy-free thus $v_i(A_i) < v_i(A_j)$ for $i \in \{1, 2\}$ and $j = \{1, 2\} \setminus i$. So we get that $v_i(A_j) > v_i(A_i) \Rightarrow \frac{v_i(M)}{2} > v_i(A_i)$, $\forall i \in \{1, 2\}$ which is a contradiction. □

1.7.2 Maximum Nash welfare

Even though we know that EF allocations do not always exist we can always find an $EF1$ allocation under additive valuations.

Proposition 3. Under additive valuations an $EF1$ allocation always exists.

Proof. In order to achieve an $EF1$ allocation we use a specific mechanism to allocate goods called the draft mechanism. In the draft mechanism we run through each agent in an arbitrary order and let him choose his most valued good that has not yet been allocated to another agent.

It is easy to check that each time an agent chooses a good, he upper-bounds every choice that is being made by other agents after him, since he gets his most valued good that is available and the agent that chooses after him will choose a good which with respect to his valuation is less than the one he already chose and this stands for every draft round. Accompanied with additivity in valuations the former arguments proves our claim for the draft mechanism. □

An ideal outcome though would be to combine $EF1$ allocations with PO. The draft mechanism cannot always achieve Pareto optimality though and the reason is that there might be a beneficial swap of goods. Below we give a counterexample.

Example. Consider the following instance of two agents and four goods.

Agents	g_1	g_2	g_3	g_4
1	(5)	4	(4)	4
2	100	(ε)	ε	(ε)

We execute the draft mechanism for this instance. Assume the arbitrary order the algorithm selects to allocate the goods is first to let agent 1 choose and then let agent 2 choose. So agent 1 chooses his most valued good which is good g_1, then agent 2 chooses either g_2, g_3 or g_4 and we return again to agent 1 to repeat this procedure until all the goods have been allocated. The resulting allocation is $A = (\{g_1, g_3\}, \{g_2, g_4\})$. This allocation is $EF1$ but not Pareto optimal since agent 2 could give goods g_2 and g_4 in return for good g_1 and everyone would have higher utility than before.

Recall example 6 from section 1.6. There we presented an allocation that combined $EF1$ and PO and we pointed out that this allocation maximizes the Nash product too. Actually this is not a coincidence as it is proved in [9], the allocation that maximizes the Nash product is both Pareto optimal and $EF1$. We are about to present the key results and proofs on Pareto optimality and $EF1$ allocations.

Definition (Maximum Nash welfare solution). *The Nash welfare of allocation A is defined by $NW(A) = \prod_{i \in N} v_i(A_i)$. Given valuations v_i, the MNW solution selects the allocation A that maximizes the Nash product $\prod_{i \in N} v_i(A_i)$ among all feasible allocations.*

$$A^{MNW} \in \underset{A \in \Pi_n(M)}{\operatorname{argmax}} NW(A)$$

We say that an allocation is the MNW allocation if it can be selected by the MNW solution.

Observe that it might be possible for every feasible allocation to have zero Nash product. In this special case we find the largest set of agents to whom we can simultaneously provide positive utility and then select an allocation to these agents which maximizes the Nash product. We denote this set by S.

Theorem 1 (Cragiannis et al [9])**.** *Every MNW allocation is $EF1$ and Pareto optimal (PO) for additive valuations over indivisible goods.*

Proof. Let A denote the MNW allocation. First we deal with the case where $NW(A) > 0$. Allocation A is PO since if it was Pareto dominated by some other allocation say A' then every agent would have been at least as happy as he was thus the Nash product would increase.

Next we are about to prove $EF1$ for A. On the contrary suppose that allocation A is not $EF1$. Then there exist agents i and j, such that i envies j after the removal of any good in A_j. Let

$g^* = \text{argmin}_{g \in A_j, v_i(g)>0} v_j(g)/v_i(g)$. g^* is well defined since if agent i envies j then he must have positive valuation for at least one good in A_j. Consider A' to be the allocation in which good g^* is allocated to agent i. We will show that $NW(A') > NW(A)$, which gives the desired contradiction.

In allocation A' we have: $v_k(A_k) = v_k(A'_k),\ \forall k \neq i, j,\ v_i(A'_i) = v_i(A_i) + v_i(g^*),\ v_j(A'_j) = v_j(A_j) - v_j(g^*)$. So what we want is the following relation to hold:

$$\frac{NW(A')}{NW(A)} > 1 \Leftrightarrow \left[1 - \frac{v_j(g^*)}{v_j(A_j)}\right]\left[1 + \frac{v_i(g^*)}{v_i(A_i)}\right] > 1 \Leftrightarrow$$

$$\left[\frac{v_j(g^*)}{v_i(g^*)}\right]\left[v_i(A_i) + v_i(g^*)\right] < v_j(A_j) \tag{1.1}$$

We can also prove that

$$\frac{v_j(g^*)}{v_i(g^*)} \leq \frac{\sum_{g \in A_j} v_j(g)}{\sum_{g \in A_j} v_i(g)} = \frac{v_j(A_j)}{v_i(A_j)} \tag{1.2}$$

Since A is not $EF1$ we know that even after removing good g^* for j to i we get

$$v_i(A_i) + v_i(g^*) < v_i(A_j). \tag{1.3}$$

Multiplying (1.2) and (1.3) gives

$$\left[\frac{v_j(g^*)}{v_i(g^*)}\right]\left[v_i(A_i) + v_i(g^*)\right] < \frac{v_j(A_j)}{v_i(A_i)} v_i(A_i) = v_j(A_j).$$

Which is (1.1). So we proved that $\frac{NW(A')}{NW(A)} > 1$.

Next consider the case where A is the MNW solution and $NW(A) = 0$. Clearly, there are some agents that do not have positive utility. Assume S is the set of agents with positive utility. From the definition of the MNW solution we know that this is the largest set of agents that the MNW can provide positive utility. This observation is crucial for the proof.

First A is PO because any other allocation where the agents are at least as happy would either allocate a good to an agent that had zero utility which is a contradiction since the MNW has already found the largest set to guarantee positive utilities. Also any swap between the other agents would increase the Nash product which is again a contradiction since A is the allocation that maximizes it.

The $EF1$ relation holds for every agent that belongs in S. This is because A is the MNW among them. For agents not in S it suffices to prove that they are $EF1$ with respect to the agents in S. Suppose for the sake of contradiction that an agent $i \in N/S$ is envious of an agent $j \in S$ up to one good. Namely there exists a good g_j s.t $v_i(A_j/g_j) > v_i(A_i) = 0$. That means there exists at least another good say g_i s.t $v_i(g_i) > 0$, but then by removing this good from j to i we provide positive utility to agent i and agent j simultaneously. That contradicts the fact that S was is the largest set of agents to which one can provide positive utility. Hence MNW is $EF1$ and PO. □

Below we give a simple proof for relation (1.2) which we used in the proof of the last theorem.

Lemma 1. Assume g^* to be the good in A_j s.t $g^* \in \text{argmin}_{g \in A_j, v_i(g)>0} v_j(g)/v_i(g)$, it holds that

$$\frac{v_j(g^*)}{v_i(g^*)} \leq \frac{\sum_{g \in A_j} v_j(g)}{\sum_{g \in A_j} v_i(g)}$$

Proof. From hypothesis we know that $\frac{v_j(g)}{v_i(g)} \geq \frac{v_j(g^*)}{v_i(g^*)}$, thus $v_j(g) \cdot v_i(g^*) \geq v_i(g) \cdot v_j(g^*)$, $\forall g \in A_j$. Summing over all goods in A_j we get: $v_i(g^*) \cdot \sum_{g \in A_j} v_j(g) \geq v_j(g^*) \cdot \sum_{g \in A_j} v_i(g) \cdot v_j(g^*) \Rightarrow \frac{v_j(g^*)}{v_i(g^*)} \leq \frac{\sum_{g \in A_j} v_j(g)}{\sum_{g \in A_j} v_i(g)}$ □

A similar theorem holds for a relaxation of $EF1$ which we mentioned above the $MEF1$ and is proved in [9]. The theorem is stated as follows.

Theorem 2 (Caragiannis et al [9])**.** *Every MNW allocation satisfies marginal envy freeness up to one good ($MEF1$) and PO for submodular valuations over indivisible goods.*

Next we present some theorems from [9] stating the importance of the MNW to fairness approximations. We omit the proofs of the following two theorems.

Theorem 3 (Caragiannis et al [9])**.** *Every MNW allocation is π_n-maximin share (MMS) for additive valuations over indivisible goods, where*

$$\pi_n = \frac{2}{1 + \sqrt{4n - 3}}$$

Further the factor π_n is tight i.e for every $n \in N$ and $\varepsilon > 0$ there exists an instance with n agents having additive valuations in which no MNW allocation is $(\pi_n + \varepsilon)$-MMS.

The MNW solution is shown to approximate $PMMS$ as well.

Theorem 4 (Caragiannis et al [9])**.** *Every MNW allocation is Φ-pairwise MMS, where Φ is the golden ratio conjucate, i.e $\Phi = \frac{\sqrt{5}-1}{2} \approx 0.618$. Further the factor Φ is tight i.e, $\forall n \in N$ and $\varepsilon > 0$ there exists an instance with n agents having additive valuations in which no MNW allocation is $(\Phi + \varepsilon)$-pairwise MMS.*

In conclusion we present the following theorem about the implications of the fairness approximate notions.

Theorem 5 (Caragiannis et al [9]). *The pairwise maximin share guarantee is implied by envy-freeness EF and implies 1/2-maximin share guarantee, envy freeness up to he least valued good (EFX) and as a direct consequence, envy-freeness up to one good ($EF1$)*

We will just give the proof that $PMMS \Rightarrow EFX$ to warm up for the next subsection.

Proof. Consider an allocation A and let it be $PMMS$. Assume that A is not EFX. That means there are two agents say i and j s.t i envies j even after removing its least valued good call this good g^*.

It holds that $v_i(A_i) < v_i(A_j)$ and since additivity is assumed $v_i(A_i \cup g^*) > v_i(A_i)$. Taking under consideration the partition $(A_i \cup g^*, A_j/g^*)$ we point out that the $PMMS_i$ is at least $v_i(A_i \cup g^*)$ which is greater than $v_i(A_i)$, but then the allocation is not $PMMS$ since i holds A_i and has less valuation than the least $PMMS$ threshold we proved. So by contradiction we proved that $PMMS$ implies EFX. □

1.7.3 Results on EFX

Even though the MNW is related with so many fairness notions, a connection with EFX has not yet been verified. Actually the question whether every instance has an EFX allocation for additive valuations is still open. Recent results on the EFX can be found in [7] and [10] where in the last paper Chaudhury et al proved that an EFX allocation always exists for three agents under additive valuation.

Among the first papers, [28] examines the existence of EFX allocations on various valuation functions and proves results on EFX in conjunction with Pareto optimality. Another contribution of this paper though is an 1/2-approximation algorithm for EFX when agents have additive valuations.

We now present existence results on EFX that rely on the *leximin* solution. The *leximin* solution maximizes the minimum utility of any agent. If there are many allocations with the same minimum, then it minimizes the second minimum utility and so on. An alternation of the *leximin* solution the *leximin*$_{++}$ solution is used as well. Below we give the algorithms for the *leximin* operator $\prec$ and the *leximin*$_{++}$ operator $\prec_{++}$.

Algorithm 1 *Leximin* and *Leximin*$_{++}$ comparison operators

1: **function** *LeximinCmp*$(A, B, (v_1 \cdots v_n))$ ▷ Return true if $A \prec B$
2: $X^A \leftarrow$ ordering of agents by increasing utility in A and arbitrary tiebreak for same utility
3: $X^B \leftarrow$ ordering of agents by increasing utility in B
4: **for** $l \in [n]$ **do** ▷ remember $[n] = \{1, 2, \cdots, n\}$
5: $i \leftarrow X^A_l$ ▷ l-th agent in X^A
6: $j \leftarrow X^B_l$ ▷ l-th agent in X^B
7: **if** $v_i(A_i) \neq v_j(B_j)$ **then**
8: **return** $v_i(A_i) < v_j(B_j)$ ▷ returns true and stops
9: **return** false

10: **function** *LeximinCmp*$_{++}(A, B, (v_1 \cdots v_n))$
11: $X^A \leftarrow$ same as in *LeximinCmp*
12: $X^B \leftarrow$ same as in *LeximinCmp*
13: **for** $l \in [n]$ **do** ▷ remember $[n] = \{1, 2, \cdots, n\}$
14: $i \leftarrow X^A_l$
15: $j \leftarrow X^B_l$
16: **if** $v_i(A_i) \neq v_j(B_j)$ **then**
17: **return** $v_i(A_i) < v_j(B_j)$ ▷ checks and returns true and stops
18: **if** $|A_i| \neq |B_j|$ **then**
19: **return** $|A_i| < |B_j|$ ▷ checks and returns true and stops
20: **return** false

What the *leximin* operator does is that for any given allocations A and B, it considers their induced utility vectors in increased order and compares the components one by one. If the relation in step 8 is satisfied then it stops and returns true that $A \prec B$. The *leximin*$_{++}$ does the same. It receives the same input as the *leximin* and makes the same computations. The key difference is when the relation in step 8 holds with equality then it compares the bundle sizes as we can see in step 22. If it holds $|A_i| < |B_j|$ then it terminates and outputs $A \prec_{++} B$.

This comparison in bundle sizes is being made actually because there are goods for which the agents have zero valuation. It might not be very clear but we will prove that when we do not allow zero valued goods in the instance the *leximin* solution is EFX for a specific kind of valuation functions, whether $leximin_{++}$ does the job when zero valuation is allowed. Below we give an example to give a glimpse that whenever zero valuation is allowed the *leximin* solution is not EFX.

Example. Consider two agents with identical valuations over two goods g_1 and g_2. Let the valuations defined as $v(g_1) = 0$, $v(g_2) = 1$, $v(\{g_1, g_2\}) = 2$. The *leximin* solution is the allocation $A' = (\{g_1, g_2\}, \emptyset)$, because the minimum utility of every allocation is zero and A' has the higher second minimum. Allocation A' is obviously not EFX since if we remove good g_1 from agent 1, agent 2 is still envious.

So the *leximin* solution fails when zero valuations are allowed. The $leximin_{++}$ on the contrary can give EFX allocations but only when the agents have identical valuations. In order to give a proof to this claim first we must prove that $leximin_{++}$ specifies a strict total ordering over the allocations.

Theorem. *The $\prec_{++}$ comparison specifies a total ordering*

Proof. To prove strict total ordering for the $leximin_{++}$ relation we must prove irreflexivity and transitivity.

First we prove that $A \prec_{++} A$ is false. Check the algorithm 1 for the same input X^A. On each iteration, the same agent is considered so $A \prec_{++} A$ never terminates until it passes through every agent in the utility vector and return false

Next we prove that if $A \prec_{++} B$ and $B \prec_{++} C$ then $A \prec_{++} C$. Assume $A \prec_{++} B$ and $B \prec_{++} C$. Let l_1, l_2, l_3 be the iterations on which $A \prec_{++} B$,$B \prec_{++} C$ and $A \prec_{++} C$ terminate. Also for $x \in \{1, 2, 3\}$ let $i_x = X^A_{l_x}$, $j_x = X^B_{l_x}$ and $k_x = X^C_{l_x}$

Since $A \prec_{++} B$ terminates on iteration l_1, we have $v(A_{i_1}) < v(B_{j_1})$ or $|A_{i_1}| < |B_{j_1}|$. Similarly, since $B \prec_{++} C$ terminates on iteration l_2, we have $v(B_{i_2}) < v(C_{j_2})$ or $|B_{j_2}| < |C_{j_2}|$.

Now we prove that $l_3 \geq min(l_1, l_2)$. Suppose $l_3 < min(l_1, l_2)$ then $A \prec_{++} B$ and $B \prec_{++} C$ do not terminate until after iteration l_3. Therefore $v(A_{i_3}) = v(B_{j_3})$, $|A_{i_3}| = |B_{j_3}|$, $v(B_{j_3}) = v(C_{k_3})$, and $|B_{j_3}| = |C_{k_3}|$. Therefore $v(A_{i_3}) = v(C_{k_3})$ and $|A_{i_3}| = |C_{k_3}|$, so $A \prec_{++} C$ could not have terminated on iteration l_3, which is a contradiction. Therefore $l_3 \geq min(l_1, l_2)$. We proceed with case analysis

Case1: $l_1 < l_2$. Since $B \prec_{++} C$ did not terminate until after iteration l_1, we have $v(B_{j_1}) = v(C_{k_1})$ and $|B_{j_1}| = |C_{k_1}|$. Therefore $v(A_{i_1}) < v(C_{k_1})$ or $|A_{i_1}| < |C_{k_1}|$. We know that $A \prec_{++} C$ cannot have terminated prior to l_1 since $l_3 \geq min(l_1, l_2) = l_1$. Therefore $A \prec_{++} C$ will terminate on iteration l_1 and return true so $A \prec_{++} C$ holds in Case 1.

Case 2: $l_2 < l_1$. This is similar. Since $A \prec_{++} B$ did not terminate until after iteration l_2, we have $v(A_{i_2}) = v(B_{j_2})$ and $|A_{i_2}| = |B_{j_2}|$. Therefore $v(A_{i_2}) < v(C_{k_2})$ or $|A_{i_2}| < |C_{k_2}$. We know that $A \prec_{++} C$ cannot have terminated prior to l_2 since $l_3 \geq min(l_1, l_2) = l_2$. Therefore $A \prec_{++} C$ will terminate on iteration l_2 and return true, so $A \prec_{++} C$ holds in Case 2.

Case 3: $l_1 = l_2$. In this case we have $i_1 = i_2$, $j_1 = j_2$ and $k_1 = k_2$. Therefore we have $v(A_{i_1}) < v(B_{j_1})$ or ($v(A_{i_1}) = v(B_{j_1})$ and $|A_{i_1}| < |B_{j_1}|$), $v(B_{j_1}) < v(C_{k_1})$ or ($v(B_{j_1}) = v(C_{k_1})$ and $|B_{j_1}| < |C_{k_1}|$). Note that $v(A_{i_1}) \leq v(B_{j_1})$ and $v(B_{j_1}) \leq v(C_{k_1})$, therefore if either $v(A_{i_1}) < v(B_{j_1})$ or $v(B_{j_1}) < v(C_{k_1})$ we have $v(A_{i_1}) < v(C_{k_1})$. We know $A \prec_{++} C$ cannot have terminated before $l_1 = l_2$ since $l_3 \geq min(l_1, l_2)$ and if $v(A_{i_1}) < v(C_{k_1})$, $A \prec_{++} C$ terminates on iteration l_1 and return true. Thus assume $v(A_{i_1}) = v(B_{j_1})$ and $v(B_{j_1}) = v(C_{k_1})$: then $|A_{i_1}| < |B_{j_1}|$ and $|B_{j_1}| < |C_{k_1}|$. Therefore $v(A_{i_1}) = v(C_{k_1})$ and $|A_{i_1}| < |C_{k_1}|$, so $A \prec_{++} C$ terminates on iteration l_1 and returns true. Therefore $A \prec_{++} C$ in Case 3. This shows that $A \prec_{++} B$, $B \prec_{++} C$ implies that $A \prec_{++} C$ and completes the proof. □

Next we prove the first theorem on the $leximin_{++}$ solution.

Theorem 6 (Plaut and Roughgarden [28]). *For general but identical valuations the $leximin_{++}$ solution is EFX.*

Proof. Let A be an allocation that is not EFX. We will show that A is not the $leximin_{++}$ solution. Since A is not EFX there exists agents i, j and $g \in A_j$ where $v(A_i) < v(A_j \setminus g)$. Then any agent with utility $\min_k v(A_k)$ must also have utility strictly less than $v(A_j \setminus g)$, so assume w.l.o.g that $i = \text{argmin}_k v(A_k)$. If there are multiple agents with minimum utility in A, let i be the one considered last in the ordering X^A according to the arbitrary tiebreak.

Define a new allocation B where $B_i = A_i \cup \{g\}$,$B_j = A_j \setminus \{g\}$, $B_k = A_k$ for every $k \notin \{i, j\}$. We will show that $A \prec_{++} B$. Let S be the set of agents appearing before i in X^A. We know i is considered last among the agents with minimum utility by assumption so S is exactly the set of agents with minimum utility other than i. Note that $i, j \notin S$.

Since the only bundles thar differ between allocations A and B are that of i and j we have $A_k = B_k$, $\forall k \in S$ thus $v(A_k) = v(B_k) = v(A_i)$. Since $v(B_j) > v(A_i)$, j must occur after every agent in S in X^B.

Because $A_i \subset B_i$ we have $v(B_i) \geq v(A_i)$. If $v(B_i) > v(A_i)$, i must occur after every agent in S in X^B, since $v(B_i) > v(B_k)$ $\forall k \in S$. If $v(B_i) = v(A_i)$, i is still considered after every agent in S according to the arbitrary tiebreak. Thus i occurs after every agent in S in X^B in either case, which shows that the first $|S|$ agents in X^B are the agents in S in the same order they appear in X^A.

Therefore the *leximin*$_{++}$ algorithm will not have terminated before reaching position $|S| + 1$ in the orderings. Let T be the set of agents appearing after i in X^A: note that $j \in T$. By assumption of the agents with minimum utility in A, i appears last in X^A. Therefore all agents after i in X^A do not have the minimum utility so $v(A_k) > v(A_i)$ $\forall k \in T$. Recall that $v(B_j) > v(A_i)$,$\forall k \in \{T \setminus \{j\}\}$,$v(B_k) = v(A_k)$, $\forall k \in T$.

We know that $X^A_{|S|+1} = i$. If $X^B_{|S|+1} = i$, we have $|A_i| < |B_i|$ so $A \prec_{++} B$ returns true. If $X^B_{|S|+1} = k$ for some $k \neq i$ then $k \in T$. Therefore $v(A_i) < v(B_k)$, so $A \prec_{++} B$ returns true in this case as well. Since $A \prec_{++} B$, A cannot be the *leximin*$_{++}$ solution. Therefore the *leximin*$_{++}$ solution must be EFX. □

Based on this theorem Plaut and Roughgarden [28] present an algorithm that outputs an EFX allocation for general valuations but only for two agents. The algorithm is based on a cut-and-choose idea from allocation problems of divisible goods.

Agent 1 computes a *leximin*$_{++}$ allocation based on his valuations only, which is the same as the *leximin*$_{++}$ solution for many agents when the valuations are identical and equal to the valuations of agent 1. The second agent then chooses the bundle he wants more and what is left is being allocated to agent 1.

This output is EFX because agent 2 receives the bundle he values the most so he is not envious and agent 1 computed the *leximin*$_{++}$ allocation based on his valuation only, which as we said is the same as if the agents had identical valuations and we know from the previous theorem that the *leximin*$_{++}$ for general but identical valuations is EFX. So agent 1 is EFX too. Below we give the algorithm.

Algorithm 2 EFX allocation for two agents with general valuations

function CUTANDCHOOSE(m, v_1, v_2)
    $(A_1, A_2) \leftarrow Leximinsolution_{++}(2, m, v_1)$
    if $v_2(A_1) \geq v_2(A_1)$ **then**
        return (A_2, A_1)
    else
        return (A_1, A_2)

Unfortunately the $leximin_{++}$ solution fails to give an EFX outcome when the valuations are additive. Below we give the counterexample.

Example. In our example we have three agents and four goods. The valuations are shown in the table below and are assumed to be additive.

Agents	g_1	g_2	g_3	g_4
1	14	3	2	1
2	7	6	4	3
3	20	0	0	0

The $leximin_{++}$ solution is allocation $A = (\{g_2, g_4\}, g_3, g_1)$. We can check that agent 2 is not EFX towards agent 1.

Let's examine the *leximin* solution again. The reason why we abandon this concept was because for zero valuations it fails to give EFX results. Observe though that the *leximin* solution is trivially Pareto optimal, because assume it is Pareto dominated by another allocation then all agents are at least as happy as before and one is strictly happier thus the relation in step 8 of *LeximinCmp* is violated and we have our contradiction. We already know that *leximin* is not EFX for identical valuations where zero utility is allowed. Moreover, it can be proved not to be EFX even under additive valuations. There is only a special case where the *leximin* solution can be proved to be EFX and PO.

Theorem 7 (Plaut and Roughgarden [28]). *For two agents with additive valuations (not nesessarily identical) with nonzero valuations, the leximin solution is EFX and PO.*

Proof. Let A be an allocation that is not EFX. Then there exist agents i, j and good $g \in A_j$ s.t $v_i(A_i) < v_i(A_j \setminus \{g\})$. W.l.o.g assume $i = 1$ and $j = 2$.

We know that $v_1(A_1) < v_1(A_2)$, so $v_1(A_1) < 1/2$. If $v_2(A_2) < v_2(A_1)$, the agents could swap bundles to increase both their utilities, so A could not be the *leximin* solution. Therefore assume $v_2(A_2) \geq v_2(A_1)$, and so $v_2(A_2) \geq 1/2$.

Define two new bundles $S_1 = A_1 \cup \{g\}$ and $S_2 = A_2 \setminus \{g\}$. Then define a new allocation B where $B_1 = \text{argmin}_{S \in \{S_1, S_2\}} v_2(S)$ and $B_2 = \text{argmax}_{S \in \{S_1, S_2\}} v_2(S)$

Since agent 2 received his favorite between S_1 and S_2 we still have $v_2(B_2) \geq 1/2$. We have $v_1(S_2) = v_1(A_2 \setminus \{g\}) > v_1(A_1)$ by our original assumption that A is not EFX and we have $v_1(S_1) = v_1(A_1 \cup \{g\}) > v_1(A_1)$ by the nonzero valuation of v_1. Therefore regardless of which bundle agent 1 receives, $v_1(B_1) > v_1(A_1)$.

Thus B has a higher minimum utility than A so A cannot be the *leximin* solution. Therefore the *leximin* solution is EFX in this setting and trivially PO. □

The biggest contribution of [28] to the EFX notion is an 1/2-approximation algorithm for additive valuations for n agents. We are about to present this algorithm but first we give some definitions and lemmas that the algorithm uses.

Definition (**c-**EFX)**.** *An allocation A is c-EFX if,* $\forall\ i, j \in N$

$$\forall g \in A_j, v_i(A_i) \geq c \cdot v_i(A_j \setminus \{g\})$$

where $0 \leq c \leq 1$

Also we will need a graph that represents the envy between agents. Consider a graph $G = (V, E)$ where $V = N$, meaning that each node models an agent and an edge $e = (i, j) \in E$ if and only if agent i envies agent j. We also inherit an idea from [23] to eliminate cycles in this graph. The idea is that whenever a cycle occurs in the envy-graph a swap of bundles exists so as to eliminate envy. We call this procedure *EliminateEnvyCycles* and we present the lemma formally below.

Lemma 2 (Lipton et al [23])**.** Let $A = (A_1, ..., A_n)$ be a c-EFX allocation with envy-graph $G = (V, E)$ where G contains a cycle. Then there exists another allocation $B = (B_1, ..., B_n)$ with envy-graph H where B is also c-EFX and H has no cycles.

We omit the proof of lemma 2 and present directly the approximation algorithm.

Algorithm 3 1/2-EFX for n agents with subadditive valuations

1: **function** $GetApxEfxAllocation(n, m, (v_1, ..., v_n))$
2: $\quad P \leftarrow [m]$ ▷ remember that $[m] = \{1, 2, ...m\}$
3: $\quad$ **for** $i \in [n]$ **do**
4: $\quad\quad A_i \leftarrow \emptyset$
5: $\quad$ **while** $P \neq \emptyset$ **do**
6: $\quad\quad g^* \leftarrow pop(P)$
7: $\quad\quad j \leftarrow FindUnenviedAgent(A_1, A_2, ..., A_n)$
8: $\quad\quad A_j \leftarrow A_j \cup \{g^*\}$
9: $\quad\quad$ **if** $\exists\ i \in [n], g \in A_j$ s.t $v_i(A_i) < \frac{1}{2} v_i(A_j \setminus \{g\})$ **then**
10: $\quad\quad\quad P \leftarrow P \cup A_i$
11: $\quad\quad\quad A_j \leftarrow A_j \setminus \{g^*\}$
12: $\quad\quad\quad A_i \leftarrow \{g^*\}$
13: $\quad\quad (A_1, A_2, ..., A_n) \leftarrow EliminateEnvyCycles(A_1, A_2, ...A_n)$
14: $\quad$ **return** $(A_1, A_2, ...A_n)$

Theorem 8 (Plaut and Roughgarden [28])**.** *Algorithm 3 gives an 1/2-EFX allocation for n agents with subadditive valuations.*

Proof. We refer to each iteration of the while loop as a round. Let A_k^l be the bundle of agent k at the beginning round l and B_k^l the bundle of agent k at round l just before *EliminateEnvyCycle*. The proof proceeds by induction on l. If the allocation at the beginning of round l is 1/2-EFX we will prove that the allocation at the beginning of round $l+1$ is 1/2-EFX.

If lines 10-12 are not executed then allocation B^l is 1/2-EFX by definition. Assume that 10-12 are executed. Then $B_k^l = A_k^l\ \forall k \neq i$. Obviously $\forall k, k' \neq i$ EFX holds since their bundles do not change and of course for every pair(k, i) EFX hold again since i has only one good.

We show now that 1/2-EFX holds between (i, k) as well. First we show that $v_i(B_i^l) > v_i(A_i^l)$. Notice that $v_i(A_i^l) < \frac{1}{2} v_i(A_j^l \cup \{g^*\} \setminus \{g\})$ since line 9 in the algorithm holds. Thus, we have

$$v_i(A_i^l) < \frac{1}{2} v_i(A_j^l \cup \{g^*\}) \leq \frac{1}{2}(v_i(A_j^l) + v_i(g^*)) \leq \frac{1}{2}(v_i(A_i^l) + v_i(g^*))$$

and therefore

$$v_i(A_i^l) < v_i(g^*) = v_i(B_i^l).$$

Now consider any other agent k, since i is $1/2$-EFX at the beginning of round l it holds that $v_i(A_i^l) \geq \frac{1}{2} v_i(A_k^l \setminus \{g\}), \forall g \in A_k^l$. Since

$$v_i(B_i^l) > v_i(A_i^l) \geq \frac{1}{2} v_i(A_k^l \setminus g) = \frac{1}{2} v_i(B_k^l \setminus \{g\})$$

we have that B^l is also $1/2$-EFX thus concluding the proof. □

The reader can see [28] for a complete and more detailed proof.

1.7.4 Some notes on the complexity of Nash social welfare

In this last subsection we are about to present some results on the complexity of the MNW. So far we know that the MNW solution has many fairness guarantees with the strongest being the $EF1+PO$ allocation outputs. However the computational results are discouraging. It is known from [22] that the problem of maximizing the Nash social welfare for indivisible goods is APX-hard when the valuations are additive. Even for additive identical valuations it is known that the problem is NP-hard; a reduction from the PARTITION can be found in [27].

Many approximation algorithms have been constructed to surpass this computational barrier. An interesting result that combines EFX allocations with MNW approximation is presented in [4].

Lemma 3 (Barman et al [4])**.** Given an instance with additive and identical valuations, let A be an EFX allocation and A' be the MNW allocation. Then

$$SW_{nash}(A) \geq \frac{1}{1.061} SW_{nash}(A')$$

This means that for positive additive and identical valuations the EFX allocation is very close to the optimal Nash allocation in terms of the Nash product. But we already know that for general and identical valuations the *leximin* solution is EFX and PO. An algorithm that outputs an EFX allocation is presented below.

Algorithm 4 Algorithm for identical valuations

function ALG-IDENTICAL($(n, m, (v))$)
    Order the goods in descending order of value
    $A \leftarrow (\emptyset, ..., \emptyset)$
    for $k \in [m]$ **do**
        $i \leftarrow \text{argmin}_{l \in [n]} v(A_k)$
        $A_i \leftarrow A_i \cup g_k$

Theorem 9 (Barman et al [4]). *Algorithm 4 gives an EFX allocation.*

Proof. We prove the claim by induction. Let A^k to be the allocation when the k-th good is allocated. We assume that this allocation is EFX and we prove that A^{k+1} is EFX too. For case $k = 1$ the claim is obvious. Assume A^k is EFX. Remember that the goods are processed in a decreasing value order and the valuation is identical for all agents. Thus $v(g_{k+1}) \leq v(g_k)$. Since agent i receives the good he has the least valued bundle in A^k. In A^{k+1} every other agent has the same bundle so they are still EFX between them. Also agent i is EFX toward every other agent since he was in A^k and now his valuation is increased.

It remains to show that every agent $l \neq i$ is EFX towards i in A^{k+1}, but in $k+1$ the minimum valued good of i is g_{k+1} and we know that $v_l(A_l^k) \geq v_l(A_i^{k+1} \setminus \{g_{k+1}\}) = v_l(A_i^k)$ but this is the equivalent definition of the EFX notion. Thus we proved that A^{k+1} is also EFX completing the proof. □

1.8 Summary

In this first chapter we have presented the main subject of study of this thesis. We presented the classic instance of the problem in the centralized model. We defined and described the valuations of the agents over the set of goods and analyse the different kind of valuation functions that have been studied in the literature.

We presented some important efficiency measures that are used to measure the economic value of an allocation. The social welfare function that we focused on, was the utilitarian social welfare function and the Nash product. In terms of efficiency we presented the notion of Pareto optimality an efficiency measure of great theoretical importance inherited from economic theory as well. Next we presented the fairness criteria we use to evaluate the fairness of an allocation. The most important one is that on envy-freeness which is proved to be ideal since it is not guaranteed in every instance. To surpass this obstacle in fairness we introduced many fairness approximations that are being currently studied by researchers in the field.

We gave a separate section of examples on every notion that we defined previously in the chapter. To illustrate the instance of the problems we provide detailed tables and comments on them as well as detailed proofs on concepts. In the final section we presented briefly some recent results on centralized fair division of indivisible goods. We chose from a variety of papers and results those that helped us understand the ideas and those that contributed to our results.

Chapter 2

Distributed fair division

2.1 Abstract

In Distributed fair division, no central authority (algorithm) to decide to whom the goods must be allocated to exists. The agents themselves negotiate on which goods to exchange so as to maximize some social welfare function. The agents actually conduct deals, on subsets of goods, to reallocate between them. The procedure starts from a random initial allocation and the agents are negotiating, based on specific objectives, until they reach some final allocation.

Another deference in this chapter is that the agents can either receive or give money to other agents. There are two models that allow money transfer in fair division. In the first, which we present in this chapter, the agents might receive of give money to others for the goods they want to get. In the second model, which we present in chapter 3, the agents do not give money but there is an outside source that gives money to them. We will refer to the first model as fair division using monetary side payments and to the second as fair division with subsidy.

In this chapter we will use utility functions. These utility functions will not be equal just to the valuation of the received bundle, as they were in chapter 1. Our purpose is to describe payment rules and mechanism schemes so as to guarantee that the agents, who negotiate freely without central interference in these mechanisms, will converge in allocations that are economic efficient and fair.

The approach of mixing allocations with payments either from or to the agents has been extensively considered in economics literature. A typical example is the rent division problem, where n

items (rooms) and a fixed rent have to be divided among n agents in an envy-free manner, [14], [32]. Compensations to the agents were first considered by Maskin [24]. Subsequent papers consider unit-demand allocation problems, where each can get at most one item, see [1]. [2] and [21] give polynomial-time algorithms that compute allocations and payments. More general models are studied by Haake et al., [18] and Meertens et al. [25].

This chapter is organized as follows: First we give the model of a distributed fair division problem. We present the basic notions and definitions of negotiations and elaborate on the structure of deals the agents can make. Next we give some theoretical results on maximizing specific social welfare functions using specific deals. We end the chapter presenting negotiations that lead to proportional and envy-free allocations as well as envy-free allocations when the agents form a social network. We also present some negotiations without side payments. Again a special example section is presented to illustrate basic notations and definitions.

2.2 The model

The basic setup of a distributed fair division instance is exactly the same as the centralized one. Since in our model the agents negotiate between them, we can imagine the system as a multiagent society of autonomous software agents forming an artificial society where each agent has as objective to increase his utility starting from an initial random allocation. The first negotiation problems of this setup used the terminology of resources instead of goods. Here we will use the terminology goods or items as we did in the first chapter. There are many ways on how communication and selection of the negotiators is implemented. In our study we do not care that much about the model used for selecting the agents to participate in a particular negotiation. Most papers assume a selection protocol presented by Smith.R.G in, [31]. We assume the same protocol without further elaboration.

2.3 Monetary side payments

As we mentioned, we want our system to converge to fair and economic efficient allocations through negotiations, for example we would like our final allocation to maximize the average utilitarian welfare of the system namely maximize SW_{util} or the egalitarian welfare SW_{egal}. Of course as in everyday ongoing negotiations none of the agents would like a decrease in their utility as an outcome of a reallocation, because negotiations are useful if everyone increases his utility. The problem that arises is how can we combine the system's high efficiency along with higher agents utility. Here is an illustrative example.

Example. Consider an instance of two agents and one good, say g with the following valuations: $v_1(\emptyset) = v_2(\emptyset) = 0$, $v_1(g) = 4$ and $v_2(g) = 7$. Also consider the initial random allocation to be $A = (g, \emptyset)$ where agent 1 holds good g. Agent 1 is happy in contrast to agent 2. The utilitarian social welfare though is $SW_{util}(A) = v_1(g) + v_2(\emptyset) = 4 + 0 = 4$ while the maximum utilitarian welfare equals 7 and is achieved by allocating the good to agent 2. We would like agent 2 to posses the good but agent 1 would never agree in passing the good, because then his utility becomes zero. Seems like the negotiation is stuck to this local optimal solution, unless agent 2 can pay money in exchange for good g.

So in order to avoid such situations, we enable the agents to use monetary side payments. Agent 2 has to pay an amount of money say p to agent 1 to acquire the good. Of course agent 1 would like any payment above 4 because then he has utility higher than 4 and also agent 2 will not give more than 7 cause then he will not benefit from the reallocation, he would have valuation 7 from the good but will owe more than 7 to agent 1. So the payment space for agent 2 is $4 < p < 7$.

To model the payment transactions among agents we use payment functions. The payment functions we are about to present, refer to the monetary side payments model, where the money are being transferred among the agents.

Definition (Payment functions). *A payment function is a function $p : \mathbb{N} \mapsto \mathbb{R}$ from the set of agents to the reals where $\sum_{i \in N} p_i = 0$ and $p(i) > 0$ means that agent i pays money while $p(i) < 0$ means that agent i receives money. $\sum_{i \in N} p_i = 0$, which means that the total amount of money does not change in the system.*

So after each negotiation there is money transaction between the agents. At any stage of the ongoing negotiations we want to know how much money the agents have so far. To model this total bill for every agent, we use functions called payment balances denoted as π_i for each agent i. A payment balance stands for the total amount of money an agent has so far. It is obvious that $\sum_{i \in N} \pi_i = 0$. After every negotiation the system reaches a new allocation of goods where the payment balances might change for some agents. An allocation A associated with a payment balance defines a state.

Definition (State). *A state of the system is a pair (A,π) where A is an allocation and π is the payment balance vector of the agents up to this allocation.*

The negotiations start from an initial random allocation of goods among the agents. Also the agents are rewarded an initial payment, denoted as π_0, for the bundle they receive in the initial allocation.

Definition (Payment scheme). *A payment function p combined with initial payments π_0 defines a payment scheme.*

Payment schemes are important in our study of distributed fair allocation of indivisible goods. In the next sections we study specific payment schemes that when imposed on agents the system converges to fair allocations. Finally having said that we assume the utility function defined on a state (A,π) for agent i to be $u_i((A,\pi)) = v_i(A) - \pi_i$. This utility function expresses the valuation of an agent in a specific allocation A with a total money π_i.

2.4 Deals

Any new state (A, π) is reached via some negotiation among agents. The tools used in these negotiations are deals conducted by agents upon some set of goods to reallocate and what payments to make. The concept is that from a given state say (A, π) a deal is made that leads the system to a new state say (A', π'). The formal definition of a deal follows naturally.

Definition. *A deal denoted as δ is a pair $\delta = (A, A')$, where A and A' are allocations and $A \neq A'$.*

The set of agents involved in a deal $\delta = (A, A')$ is denoted as A^δ. Formally $A^\delta = \{i \in N | A_i \neq A'_i\}$ are the agents whose bundles change. For example $\delta_1 = (A, A')$ and $\delta_2 = (A', A'')$ where $A \neq A'$ and $A' \neq A''$ are both deals. Check also that $\delta_3 = (A, A'')$ would also be a deal if $A \neq A''$. If δ_3 is a deal we say that it is the composition of δ_1 and δ_2 denoted as $\delta_1 \circ \delta_2$. Furthermore if $A^{\delta_1} \cap A^{\delta_2} = \emptyset$ then we say that deal δ_3 is independently decomposable.

Definition (**Independently decomposable deals**). *A deal δ is called independently decomposable if and only if there exist deals δ_1 and δ_2 such that $\delta = \delta_1 \circ \delta_2$ and $A^{\delta_1} \cap A^{\delta_2} = \emptyset$.*

Note that when agents conduct an independently decomposable deal $\delta = (A, A')$, there exists an allocation B different from both A, A'' such that $\{i \in N | A_i \neq B_i\} \cup \{i \in N | A'_i \neq B_i\} = N$. A deal could be any reallocation of goods. The agents though are interested only in reallocations that increase their utilities. As we mentioned before no agent would agree on a deal that leads to a new state where his utility is decreased. There is no reason to participate in such deals since he has nothing to gain. The agents are only interested in individually rational deals (IR).

Definition (**IR deals**). *A deal $\delta = (A, A')$ is called individually rational denoted as IR if and only if there exist a payment function p such that $v_i(A'_i) - v_i(A_i) > p_i$*
$\forall\ i \in N$ except possibly $p_i = 0$ for agents with $A_i = A'_i$.

The above inequality means that an agent would accept a deal if and only if his gain in valuation is more than what he pays or his loss in valuation is less than what he earns. This makes sense since in that way he is only benefited from the deal.

Our desire is that any IR deal would increase the system's efficiency as well. We want for example the deals to impose an increase in the average utility of the system. Namely we want the deals to increase the utilitarian social welfare of our system which is something expected since every agent's utility is increased. Remember that $SW_{util} = \sum_{i \in N} u_i(A_i)$. Since we have states instead of just allocations, SW_{util} is defined as $SW_{util} = \sum_{i \in N} u_i(A_i, \pi_i) = \sum_{i \in N}(v_i(A_i) - \pi_i) = \sum_{i \in N} v_i(A_i) - \sum_{i \in N} \pi_i = \sum_{i \in N} v_i(A_i) - 0 = \sum_{i \in N} v_i(A_i)$.

Observe that payments do not affect the SW_{util}. This is crucial since we can continue to compute the utilitarian social welfare knowing only the allocation of a state. Since IR deals increase each agents utilities an increase in the system's utilitarian welfare is imposed too. Next we prove a very useful lemma about IR deals and SW_{util}.

Lemma 4. A deal $\delta = (A, A')$ is IR if and only if $SW_{util}(A) < SW_{util}(A')$

Proof. $' \Rightarrow'$: Assume an IR deal $\delta = (A, A')$. By definition an IR deal exists if and only if there exists a payment function p s.t $v_i(A'_i) - v_i(A_i) > p_i\ \forall\ i \in N$ except possible $p_i = 0$ if $A_i = A'_i$. Summing up the inequality for all agents $i \in N$ we get: $\sum_{i \in N}(v_i(A'_i) - v_i(A_i)) > \sum_{i \in N} p_i$. Recall by definition that in any state $\sum_{i \in N} p_i = 0$. So finally we get that $SW_{util}(A') = \sum_{i \in N} v_i(A'_i) > \sum_{i \in N} v_i(A_i) = SW_{util}(A)$. Hence we get our claim that $SW_{util}(A') > SW_{util}(A)$.

$' \Leftarrow '$: Let's assume now that $SW_{util}(A) < SW_{util}(A')$ and prove that deal $\delta(A, A')$ is IR. For a deal to be IR there must exist a payment function to satisfy $v_i(A'_i) - v_i(A_i) > p_i \; \forall i \in N$. In order to prove our claim we must find such a payment function. Consider the payment function $p_i = v_i(A'_i) - v_i(A_i) - \frac{SW_{util}(A') - SW_{util}(A)}{|N|}$, $\forall i \in N$. First it is easy to check that p is actually a payment function, if we sum for all agents we get $\sum_{i \in N}((v_i(A'_i) - v_i(A'_i)) - |N| \frac{SW_{util}(A') - SW_{util}(A)}{|N|} = SW_{util}(A') - SW_{util}(A) - SW_{util}(A') + SW_{util}(A) = 0$. Also this specific payment function satisfies the inequality in the definition of IR deal : $v_i(A'_i) - v_i(A_i) > v_i(A'_i) - v_i(A_i) - \frac{SW_{util}(A') - SW_{util}(A)}{|N|} \Rightarrow SW_{util}(A') > SW_{util}(A)$ which is true as our assumption. □

Lemma 4 states that the utilitarian social welfare is an appropriate measure for the system's efficiency. Why is that? Because the agents are interested in IR deals only and lemma 4 states that IR deals increase the SW_{util}. Moreover due to this increase in the system's SW_{util}, it is possible after a number of IR deals (or maybe one) that the agents will reach a state where the allocation gives to the system its maximum utilitarian welfare. From now on we will refer to this allocation as the efficient allocation to discriminate among others.

Definition (Efficient allocation). *We call the allocation A which maximizes the system's SW_{util} the efficient allocation. Also we call a state (A, π) efficient if and only if allocation A is an efficient allocation.*

Having proved lemma 4, someone might think that the system can reach or must reach the efficient allocation if the agents conduct only IR deals. At each new state the SW_{util} increases and every agent's utility increases. If we take under consideration the fact that the number of all possible allocations is finite, $(|M|^{|N|})$, eventually every IR deal sequence will converge to the efficient allocation.

What we do not take under consideration though is what exactly the structure of a deal is. What can we negotiate in a deal, can we only give one good at the time, only a set of goods from an agent to another, are we allowed only to swap goods or are we allowed to exchange whatever we want with whoever we want. The structure of the deal is very important because sometimes it might be the case that we cannot reach the efficient allocation because the structure of the deal does not allow us to make the transition from the current state to the efficient. Below we give an illustrative example.

Example 7. Consider the following setup.

Agents	g_1	g_2	g_3	g_4	M
1	5	5	0	0	10
2	(4)	(4)	(4)	(4)	18
3	0	0	5	5	10

We have three agents and four goods. The valuations are given in the above table. Assume that the valuations for the bundles not shown in the table are additive. So agents 1 and 3 have additive valuations and agent's 2 valuation is supermodular. Suppose in the initial allocation agent 2 holds all the goods, thus his valuation is 18 and the $SW_{util} = 18$. Observe that this is not the efficient allocation because if agents 1 and 2 hold the goods that they value positively, then $SW_{util} = 20$.

Agents	g_1	g_2	g_3	g_4	M
1	(5)	(5)	0	0	10
2	4	4	4	4	18
3	0	0	(5)	(5)	10

Notice that the only way to jump to the efficient allocation is by passing two goods from agent 2 to agent 1 and 3 but at the same deal. If agent 2 passes both two goods to agent 1 or 3 in one deal then $SW_{util} = 18$ and from lemma 4 this is not an IR deal. So the structure of the deal must allow the transition of a set of goods to more than one agents and at the same time.

2.5 Deal types and structures

The structure and form of the deals have been studied extensively by Tuomas W. Sandholm in [30]. Our presentation of various deal types differs a bit from the original study in [30]. We adapt the study to our notations and definitions for terms of continuity without changing the main ideas and results. The analysis below is based on the monetary side payments model and not on the subsidy model. In every case below money is being transferred from an agent to another. In [30] the author uses the term contracts instead of deals and the term tasks instead of goods. We maintain the terms deals and goods. So far we were only interested in IR deals, deals that increase SW_{util}, and the transition made from a state to another after a deal is conducted. There are many kinds of deals one can discriminate in every day ongoing negotiations.

Deals concerning only one good and thus two agents which we call O-deals (O-contracts in the original paper), deals concerning more than one good and again two agents which are called as C-deals (C-contracts), deals where two agents agree on swapping goods called S-deals (S-contracts), deals where three or more agents participate and give or receive one good which are referred to as M-deals (M-contract) and finally deals where every agent can either receive or give more than one good from or to more than one agent. The last ones are actually a combination of all the above deal types and are called OCSM-deals.

Our main interest is whether some kind of these deals is sufficient to lead the negotiations eventually to the efficient state. We analyse each type of deals and answer this question for each kind separately.

2.5.1 O-deals

Many negotiations are about one good that an agent posses and another agent wants. The deals used in these situations are called O-deals or 1-deals.

Definition (O-deals). *An O-deal or 1-deal is a pair* $(A_{i,j}, p_{i,j})$ *where* $|A_{i,j}| = 1$ *denotes the good that is transferred from agent* i *to agent* j *and* $p_{i,j}$ *the money agent* i *receives from agent* j *in order to give him the good.*

Informally an O-deal is a single good sale from an agent to another. Since the valuations are private information of the agents, it is compatible to assume that the agent who wants a good claims it from the agent that posses it. The question is can the system, using only O-deals, reach the efficient state? Before answering the question we will present a useful model for studying each of these deal types.

Key Analysis Model: From now on w.l.o.g when we do not care about the payment balances, we will refer to allocation A instead of state (A, π). It is very useful to model the transition from an allocation to another after a conducted deal using a graph $G = (V, E)$. Its vertices are all the possible allocations and an edge connects two allocations A, A' if and only if there exists a deal $\delta = (A, A')$ of the specific type we examine. The set of edges depends on the kind of deals we allow the system to use.

Example. Given two agents and two goods say g_1 and g_2, the vertices of the induced graph are $a = (\{g_1, g_2\}, \emptyset), b = (\{g_1\}, \{g_2\}), c = (\{g_2\}, \{g_1\}), d = (\emptyset, \{g_1, g_2\})$. If we allow only O-deals then the set of edges is $E = \{ab, ac, cd, bd\}$. Since an O-deal allows the transfer of only one good, jumping from allocation a to d using O-deal is impossible, the agents cannot agree on passing more than two goods at one deal.

From now on we use this graph model in our analysis. To return to O-deals, what we pointed out in the last example is very important, there is no O-deal to lead the system from allocation a to allocation d. Of course there is a path in the induced graph from a to d due to the fact that the graph is connected.

Now consider the scenario where the agents start negotiating from the initial allocation a and the efficient allocation is d and the agents are allowed to conduct only O-deals. The are two paths from a to d: abd and acd. So there are two sequences of O-deals to lead the system to the efficient allocation. Remember though that the agents agree only on IR deals. So what happens if none ab and ac deals are IR? The answer is that the system will stuck in a suboptimal solution, that is allocation a.

If someone read example 7 again, he will notice that this is exactly what happens to the system when the initial allocation gives all goods to agent 2 and then allows only O-deals. Every O-deal is not IR and the only way to jump to the efficient allocation was to pass at the same deal g_1 and g_2 to agent 1 and g_3 and g_4 to agent 3 which is not an O-deal.

Someone might notice also that in example 7 the valuations are not additive for everyone, agent 2 has supermodular valuations and this supermodularity keeps the system away from converging to an efficient allocation. So what if agent 2 had also additive valuation? It can be proved that when agents have additive valuations a path to the efficient allocation from any initial allocation always exists.

Proposition 4. If all valuation functions are additive, any sequence of IR O-deals will eventually converge to the efficient allocation.

Proof. Firstly it is possible from any initial allocation to reach the final efficient allocation, just transfer goods one at a time until the efficient allocation is formed. Since the deals are IR any new reached allocation has higher SW_{util} as we proved in Lemma 4. Assume that the system is at a state (A, π). We define a characteristic function f_A that maps any good to the agent who possesses it in allocation A. For example $f_A(g) = i$ means that agent i holds good g in allocation A. $SW_{util}(A)$ can be now defined as $SW_{util}(A) = \sum_{g \in M} u_{f_A(g)}(g)$.

Now suppose that the negotiations have terminated so no other IR deal exists and assume also that A is not the efficient allocation. Then another allocation exists with higher SW_{util}, say allocation B. Thus there must be at least one good g such that $u_{f_A(g)}(g) < u_{f_B(g)}(g)$ that means an IR O-deal exists that passes the good from $f_A(g)$ to $f_B(g)$ but that is a contradiction since we assumed that the negotiation has terminated. So the final reached allocation must be efficient. □

So in additive scenarios O-deals are sufficient to converge the system to efficient allocations. Below we give a very simple example that demonstrates this.

Example O. Consider one good g and two agents with valuations $v_1(g) = 2$ and $v_2(g) = 1$. Let the system begin form the initial allocation $A = (\emptyset, g)$. The O-deal that transfers good g from agent 2 to agent 1 increases the SW_{util} and leads to the efficient allocation.

We end the O-deal conversation keeping in mind two important notices. First the induced graph when we use O-deals is connected. There is always a path leading from an allocation to another (just move one good at the time). The connectivity of these graphs is a necessary condition if we want to converge to efficient allocations, before start worrying about IR deals.

Think a scenario where the kind of deals we allow induce a graph that is not connected. Take then two vertices that are not connected and assume one is the initial allocation and the other is the efficient. There is no way to reach the efficient allocation from the initial one even if all the deals from the initial allocation are IR. This is a structual constraint the form of the deal imposes.

2.5.2 C-deals

Now we study deals that allow agents to transfer a set of goods instead of just one. These deals are called C-deals or cluster-deals (C-contracts originally).

Definition (C-deals). *A C-deal is a pair* $(A_{i,j}, p_{i,j})$ *where* $|A_{i,j}| > 1$ *denotes the set of goods transferred from agent* i *to agent* j *and* $p_{i,j}$ *the money agent* i *receives from agent* j.

It is easy to notice that the induced set of edges in a graph where we allow only C-deals is disjoint from the induced set when we allow O-deals. This means that given an allocation say A the set of all possible allocations that are reached via a single C-deal from A is disjoint from the set of all possible allocations reached from A via an O-deal. Here is an example of a C-deal leading to the efficient allocation.

Example C. Consider two agents and two goods with valuations $v_1(g_1) = v_1(g_2) = 1$, $v_1(\{g_1, g_2\}) = 3$ and $v_2(g_1) = v_2(g_2) = 1$, $v_2(\{g_1, g_2\}) = 4$. Assume the initial allocation $A = (\{g_1, g_2\}, \emptyset)$, where $SW_{util}(A) = 3$. The only allocation that increases the SW_{util} is the one where agent 2 holds both goods. Allowing only C-deals, the deal after which both goods are allocated to agent 2 leads the system to the efficient allocation.

Are C-deals alone sufficient to lead the system always to the efficient allocation from any random initial allocation? The answer is again no, because we can construct an instance whose induced graph allowing only C-deals is not even connected.

Example. Any instance with two agents and one good can never reach the efficient allocation using only C-deals. Actually in a situation like this no C-deal can be conducted at all.

2.5.3 S-deals

Next we discuss the class of S-deals or swap deals (S-contracts originally). This structure of deals allows two agents to swap only two goods between them.

Definition (S-deals). *A S-deal is a 4-tuple* $(A_{i,j}, A_{j,i}, p_{i,j}, p_{j,i})$ *where* $|A_{i,j}| = |A_{j,i}| = 1$ *denotes the good the agents receive from the other one and* $p_{i,j}$ *are the money agent* j *pays agent* i *while* $p_{j,i}$ *are the money agent* i *pays agent* j.

Again the induced set of edges in a graph where only S-deals are allowed is disjoint from both edge sets in O-deal and C-deal graphs. That means that there are paths in this graph that are not in the above ones and vice versa. Here is an example of a S-deal leading to the efficient allocation

Example S. Consider two agents and two goods with valuations $v_1(g_1) = 1, v_1(g_2) = 2, v_1(\{g_1, g_2\}) = 2$ and $v_2(g_1) = 2, v_2(g_2) = 1, v_2(\{g_1, g_2\}) = 2$. Assume the initial allocation is $A = (\{g_1\}, \{g_2\})$ with $SW_{util}(A) = 2$. The efficient allocation is when agent 1 and 2 swap goods g_1 and g_2 with $SW_{util} = 4$.

Again we can find an example that S-deals alone are not sufficient to lead the system to maximum efficiency. Thus the induced graph when we use only S-deals is not always connected either.

Example. Obviously any instance with two agents and one good starting from any suboptimal allocation can be used as an example. Notice also that in swap-deals the agents always have the same number of goods in their bundle as in the initial allocation because for every good they give, they receive one. If in the optimal allocation their bundle has more or less goods then no sequence of swap-deals can guarantee that bundle.

2.5.4 M-deals

All kinds of deals we presented so far have been proved to not always induce connected graphs thus we cannot expect using only one kind of these deals to always converge to an efficient allocation. Next we present M-deals or Multiagent-deals (originally M-contracts).

Definition (M-deals). *A M-deal is a pair $(\boldsymbol{A}, \boldsymbol{p})$ of $|N| \times |N|$ matrices (recall that N denotes the set of agents), where at least three elements of $\boldsymbol{A}$ are non-empty. An element $A_{i,j}$ of the A matrix is a good agent j receives from agent i and for all i and j $|A_{i,j}| \leq 1$. $p_{i,j}$ is an element of p matrix and stands for the amount of money agent j pays agent i.*

M-deals seem to be more flexible than the previous kinds of deals. First they engage more than two agents at one deal which none of the other deals allowed and secondly they allow every agent to receive at most one good from another agent at the same time. Below we give an example of how a M-deal can lead the agents to the efficient allocation.

Example M. Consider three agents and three goods g_1, g_2, g_3 with valuations:

Agents	g_1	g_2	g_3	$\{g_i, g_j\}$	$\{g_1, g_2, g_3\}$
1	(4)	1	5	0	0
2	5	(4)	1	0	0
3	1	5	(4)	0	0

Assume that the agents start negotiating from allocation $A = (g_1, g_2, g_3)$ with $SW_{util}(A) = 12$. The M-deal in which agent 1 receives good g_3, agent 2 receives good g_1, and agent 3 receives good g_2, lead to the efficient allocation with $SW_{util} = 15$.

Agents	g_1	g_2	g_3	$\{g_i, g_j\}$	$\{g_1, g_2, g_3\}$
1	4	1	(5)	0	0
2	(5)	4	1	0	0
3	1	(5)	4	0	0

Again there exist examples where M-deals alone are not sufficient to find the efficient allocation. Of course what stands for the edge set of the previous induced graphs stands for M-deals too.

Example. Any instance of 2 agents suffices to prove that no M-deal can lead the system to the efficient solution. Actually no M-deals even exist by definition in those instances.

2.5.5 OCSM-deals

We analysed O, C, S, M deals and presented examples where none of them alone is sufficient enough to use exclusively, in order to reach the efficient allocation. Remember again that our goal is for the system to be able to always reach efficient allocations.

What rises from the remarks we have made so far, is that all these deals induce disjoint sets of edges. That means that there might be paths through allocations or in other words a negotiation sequence, that depend on the kind of deals we allow. It is possible that a path to the efficient allocation using for example only C-deals exists while using only S-deals does not or vice versa.

If the negotiations are stuck in a local optimum solution, perhaps it might be possible to jump to another allocation if we were allowed to use other deal structures. If we study again Example

O, Example C, Example S and Example M we can observe that O-deals are the only kind of deals that can lead to the efficient allocation in Example O, C-deals are the only kind of deals that can lead to the efficient allocation in Example C, S-deals are the only kind of deals that can lead to the efficient allocation in Example S and M-deals are the only kind of deals that can lead to the efficient allocation in Example M. What is stated is that there is not an exclusive choice of deal types to use alone which always gives converges guarantees. Namely if we pick a deal type, say for example O-deals and give the instance of Example S, we can never converge to the efficient allocation.

So what about letting the agents to use all 4 kinds of deals in their negotiations? At any time an agent or a set of agents might conduct an O or C or S or M deal. In this way the induced graph will have more edges, which means more negotiation paths, because the system can now jump from an allocation to many more than using less than these 4 deal types. Sadly this idea fails too. Below we give a negative example.

Example OCSM. Consider an instance consisting of three goods and two agents with valuations:

Agents	g_1	g_2	g_3	$\{g_1, g_2\}$
1	①	①	3	2
2	2	2	③	4

Also consider that the valuations of bundles not shown in the table are 0. Assume the initial allocation $A = (\{g_1, g_2\}, g_3)$ where $SW_{util}(A) = 5$. It easy to check that the only deal that leads to the efficient allocation $A' = (g_3, \{g_1, g_2\})$ with $SW_{util}(A') = 7$ is the deal that transfers goods g_1, g_2 to agent 2 and good g_1 to agent at the same.

Agents	g_1	g_2	g_3	$\{g_1, g_2\}$
1	1	1	③	2
2	②	②	3	4

Does this deal belong in any deal type we have already mentioned? The answer is no.

Of course even if we allow any combination of three or two kind of deal types we will not achieve something better. Think that by reducing the kind of deals we remove edges from the induced graph thus making it harder not only to be just connected but also connected via IR deals. In the last example though the deal we wanted so as to jump to the efficient allocation can be decomposed into two deals, one is the C-deal for passing $\{g_1, g_2\}$ to agent 2 and the other is passing g_3 to agent 1 which is an O-deal, but the fact that we do not allow these deals to happen simultaneously forces the system to stuck in a suboptimal allocation. Is there a deal structure to achieve this simultaneous combination of the former deals? The deal structures we are about to define now meets this desired property.

Definition (OCSM-deals). *An OCSM-deal is a pair* $(\boldsymbol{A}, \boldsymbol{p})$ *of* $|N| \times |N|$ *matrices. An element* $A_{i,j}$ *of the A matrix is a good agent* j *receives from agent* i *and* $p_{i,j}$ *is an element of* p *matrix and stands for the amount of money agent* j *pays agent* i*.*

In an OCSM-deal the agents can receive or give at least one good and at least two agents can participate. Observe that any O, C, S, M - deal is an OCSM-deal too. This implies that the union of all sets of edges induced by the kinds we have so far analyse is a strict subset of the set of edges the induced graph has, when agents agree using OCSM-deals.

We already presented a deal in Example OCSM that is OCSM and does not belong to the any other deal type, thus justifying the strict subset relation. The question again is, does a sequence of IR OCSM-deals that lead to the efficient allocation always exists? We prove this in the following theorem.

Theorem 10. *Given a distributed allocation of indivisible goods instance there always exists a sequence of IR OCSM-deals that converges to the efficient allocation.*

Proof. Assume a distributed allocation of indivisible goods instance. To prove our claim first we must prove that the induced graph of this instance is connected. Consider any two random allocations say A and A' and assume in A' an agent might receive some goods from some other agents and might give some goods to others. The OCSM-deal definition does not put any constraint on the numbers of goods he can give or receive and on the set of participants. It is easy to form a matrix $\mathbf{A}$ for each agent j, just go and add a 1 to the i-th row if and only if agent i gives a good to j in A'.

So the deal that is conducted to jump from A to A' satisfies the definition of an OCSM-deal. That means an OCSM-deal from A to A' does exist. Of course this holds for any two random allocations, thus the induced graph is not only connected but fully connected, since from any given allocation an OCSM-deal exists to make the transition to any other allocation. So OCSM-deals meet the necessary condition we want.

Remember that the agents agree only on IR deals. In Lemma 4 we proved that a deal is IR if and only if the allocation it leads to has higher SW_{util} than the current allocation. Since the deals are IR we know that any reached allocation has higher SW_{util}, meaning that the deals are forming a path in the induced graph where the allocations are passed through in increased order. Since the number of allocations is finite this path cannot be infinite. In the worst case the agents will pass through every allocation before reaching a final one and no bactracking exists because this passing is in increased SW_{util} order.

Now suppose the system reaches an allocation E and no other IR deal is possible. If E is not the efficient allocation then there exists another allocation say B with higher SW_{util}, but then from Lemma 4 there exists an IR deal from E to B but this contradicts the fact that no other IR deal exists as we assumed. So the final allocation reached by the agents must be the efficient. □

2.6 GUPF - LUPF

Now that we established the basic theory on deal types we return to the conversation about payment functions. Remember that the agents in this model use payments to pay for the goods they receive and also the total sum of payments is always zero. Moreover there is a payment balance π for each agent. Sometimes we give extra money to the agents in the initial allocation to reward them. The combination of these initial payments π_0 with a payment function is called a payment scheme.

Of course the payments the agents make may be arbitrary, namely it is not always the case that there would be an explicit formula for them. Notice also that the payment functions do not give information about who pays who, rather that some agents give money (if $p_i > 0$) and some others receive ($p_i < 0$). In this section we present some specific payment functions introduced by Estivie et al at [16]. We know that any IR deal from an allocation A to A' increases the SW_{util} of the system. The payment functions we introduce now essentially redistribute the social surplus $SW_{util}(A') - SW_{util}(A)$.

Definition (LUPF). *The locally uniform payment function (LUPF) is defined as* $p_i = [v_i(A') - v_i(A)] - \frac{[SW_{util}(A') - SW_{util}(A)]}{|A^\delta|}$*, if* $i \in A^\delta$.

The LUPF payment function divides equally the social surplus of an IR deal among the agents that participate in that deal. The payment function GUPF (globally uniform payment function) on the contrary divides the social surplus among all agents in the system.

Definition (GUPF). *The globally uniform payment function (GUPF) is defined as* $p_i = [v_i(A') - v_i(A)] - \frac{SW_{util}(A') - SW_{util}(A)}{|M|}$.

Both LUPF and GUPF are payment functions, it suffices to add p_i over all agents $i \in N$ and check that the total sum is 0. The reason for mentioning these functions is because they play an important role as we will see in helping the system to converge to fair and proportional allocations.

2.7 Fairness notions

Our main interest as stated from the beginning is to find allocations to the agents that meet specific fairness notions. In the chapter 1 we presented many fairness notions like $Proportionality$ and EF, $EF1$, EFX, MMS and more. Now we give the definitions of proportionality and envy-freeness in a different way but the main idea behind these definitions is the same. We will talk about envy-freeness and proportionality of a state (A, π) instead of just allocation A. Let's consider an instance consisting of a set N of n agents and a set M and m goods.

Definition (Proportionality). *A negotiation state* (A, π) *is called proportional if and only if* $u_i(A_i, \pi_i) \geq v_i(M)/n$, $\forall i \in N$.

For additive valuation functions, proportionality is a desired fairness notion. We would like our system to converge to a proportional state that has high efficiency too. Next we demonstrate a distributed procedure for obtaining an allocation that is both efficient and proportional. For efficiency we already know that using IR OCSM-deals, the system always finds a path that leads to the efficient state. Since we have this information we let the agents conduct only these type of deals. The question is how can we guarantee proportionality.

Of course the efficient allocation is not always proportional but since the IR deals will lead to this allocation, are there any payment functions or even better payment schemes to guarantee proportionality? The answer is yes. The special payment scheme we will force the agents to adapt is called Knaster payment scheme. Before analysing this payment scheme we will introduce a method for finding a proportional allocation for the system assuming it is not distributed.

2.7.1 The Knaster procedure

The Knaster Procedure is a classical proportionality procedure produced by Bronislaw Knaster. One can find more detailed information about the procedure at [29]. The Knaster procedure is an algorithm for computing a proportional state as follows:

- Compute an allocation A^* that maximises the utilitarian social welfare of the system (efficient allocation).
- For every agent in N compute his excess, $ex_i(A^*) = v_i(A^*) - v_i(M)/n$. Compute the total excess of the system $EX(A^*) = \sum_{i \in N} ex_i(A^*)$.
- Impose on every agent a payment $p_i = ex_i(A^*) - EX(A^*)/n \; \forall i \in N$.

The payment p we impose on the agents is indeed a payment function, check that $\sum_{i \in N} p_i = \sum_{i \in N}[ex_i(A^*) - Ex(A^*)/n] = Ex(A^*) - Ex(A^*) = 0$. After executing the Knaster procedure we would have an allocation A^* which is efficient and each agent has $u_i(A^*, \pi) = v_i(A^*) - [ex_i(A^*) - Ex(A^*)/n] = v_i(M)/n + Ex(A^*)/n$, where $Ex(A^*) = SW_{util}(A^*) - \frac{1}{n} \cdot \sum_{i \in N} v_i(M)$ which is easy to check that it is positive. So each agent has eventually $u_i(A^*, \pi) = v_i(M)/n + Ex(A^*)/n \Rightarrow u_i(A^*, \pi) \geq v_i(M)/n$ which is the definition of proportionality, thus A^* is proved to be proportional.

2.7.2 Knaster payment scheme

The Knaster procedure actually has to do with centralized fair division with payments. We want to transfer this idea to the distributed setup. The IR deals are again imposed on the agents so as to have the efficient allocation as an outcome of the negotiations and a specific payment scheme will be imposed also so as to have the desired proportional outcome. We will form a mechanism where independently of the negotiations, the system will eventually converge to a proportional and efficient allocation.

In a distributed instance where the agents negotiate between them, we know that the initial allocation is random, so it is possible that the initial allocation is already the efficient allocation. In that case we would want immediately the payments to be Knaster. Being so we would have an efficient allocation at the beginning combined with Knaster payments which as we already proved outputs a proportional allocation. So it is necessary to provide the agents with initial Knaster payments, in case the initial random allocation is the efficient. Assume A_0 is the initial allocation, then the initial payments must be $\pi_0 = ex_i(A_0) - Ex(A_0)/n$.

The reason why we impose the Knaster payments is because we do not know a-priori whether the initial allocation is the efficient or not. If the initial is not the efficient then the second allocation might be or the third. Remember that we do not know when the efficient allocation will be reached, but we know that after a finite number of deals the efficient allocation eventually will be reached thus we want at any state the total payments for every agent to be Knaster. That means at any state say (A, π) we want $\pi_i = ex_i(A) - Ex(A)/n$, $\forall i \in N$.

So we know the initial payments and the total payments at any state. What we need to define the payment scheme is the payment function. Notice that in a deal say $\delta = (A, A')$, we know that $p_i = \pi_i(A') - \pi_i(A)$ and also π_i is Knaster. So finally the desired payment function is $p_i = ex_i(A') - Ex(A')/n - ex_i(A) + Ex(A)/n = [v_i(A') - v_i(A)] - [SW_{util}(A') - SW_{util}(A)]/n$. This payment function is the GUPF.

To conclude we have proved which initial payments and payment functions, namely payment scheme we need so as to ensure proportional and efficient outcomes. A more detailed presentation can be find in [12]

Definition. *We call the GUPF combined with initial payments the Knaster payments as the Knaster payment scheme.*

The theorem for finding a proportional allocation presented by Chevaleyre et al at [12], follows naturally.

Theorem 11 (Chevaleyre et al [12])**.** *Under the Knaster payment scheme any sequence of IR deals will always converge to a proportional state.*

2.8 Envy-free states

The most important fairness criterion is envy-freeness. We gave the definition of an envy-free (EF) allocation. It is not always possible to find an EF allocation, actually one might not even exist but when using money we will prove that one EF state always exists. Money help the system eliminate envy. Our goal in this section is to find whether a payment scheme that helps the system to always converge to an efficient envy-free state exists. First we define an EEF state.

Definition (EF **states**). *A negotiation state* (A, π) *is called envy-free or* EF *if and only if* $u_i(A_i, \pi_i) \geq u_i(A_j, \pi_j)$, $\forall i, j \in N$. *An* EF *state that is both efficient is called* EEF *state.*

The EEF notation is the same as the epistemic envy-freeness approximation we gave at chapter 1. Throughout this chapter when we refer to EEF we mean an efficient envy-free state. To understand the above relation and why money favour envy-freeness remember the example we gave at the beginning of section 2.3. Assume that agent 2 holds good g, then $u_2(g) = 7$ and agent 1 has zero utility so he is jealous of agent 2. If no money were allowed then this allocation could not become envy-free. With money though agent 2 can give say 3.5 units of money to agent 1 then $u_1(\emptyset, \pi_1) = 3.5$ and for agent 2 we have $u_2(g, \pi_2) = 7 - 3.5 = 3.5$. Now agent 1 is not envious since $u_1(A, \pi_1) = 3.5 \geq u_1(g, \pi_2) = 4 - 3.5 = 0.5$ and agent 2 is not envious as well because $u_2(g, \pi_1) = 7 - 3.5 = 3.5 \geq u_1(\emptyset, \pi_2) = 3.5$.

We want EEF sates when the system is distributed, meaning while the agents are conducting IR deals we would like to have guarantees that there are states that are EF. Of course not all states can become EF using money. The relation we want to satisfy in order to have an EF state induces a number of constraints that must hold in order for the state to become EF with payments. These constraints form an LP which we use to exact information about the enviousness in a given state.

2.8.1 LP, Envy-graph, positive cycles

We have defined that for a state (A, π) to be EF it must hold:

$$u_i(A(i), \pi_i)) \geq u_i(A(j), \pi_j), \forall i, j \in N, i \neq j \tag{2.1}$$

Analysing this relation we get the following constraints.

$$\pi_j - \pi_i \geq v_i(A(j)) - v_i(A(i)), \forall i, j \in N, i \neq j \tag{2.2}$$

All these constraints are linear in terms of π_i and we also have that $\sum_{i \in N} \pi_i = 0$. So we can form an LP whose solution outputs as an answer whether there are payments to make the given state EF or in other words whether the state is envy-freeable. We remind again that this is the LP in the monetary side payments model. The LP is as follows:

$$min \quad \{\emptyset\}$$

subject to:

$$\pi_j - \pi_i \geq v_i(A(j)) - v_i(A(i))$$

$$\sum_{i=1}^{n} \pi_i = 0$$

$$\forall i, j \in N, i \neq j$$

Given an allocation A we can construct the envy-graph of this allocation which is a complete directed graph $G(V, E)$ where V $= \{i \mid i \in N\}$ and E $= \{(i, j) \mid i, j \in N\}$. Furthermore $\forall$ $(i, j) \in E$ we assign the value $v_i(A(j)) - v_i(A(i))$. Below we give some definitions we are about to use to make connections between the Envy-graph and the LP stated above.

Definition. *Given the Envy-graph of an allocation A, we say that the envy graph contains a positive cycle if and only if there is a cycle in the Envy-graph whose total sum of the values assigned to his edges is positive. Namely* $\exists$ $C \subseteq E$ *s.t* C *is a cycle and* $\sum_{e \in C} wgt(e) > 0$

Actually this is a condition that shows whether an allocation is envy-freeable. If the Envy-graph of an allocation contains a positive cycle then the allocation cannot be envy-freeable. This claim is proved below:

Proposition 5. Given a state (A, π) and its induced Envy-graph, if the Envy-graph contains a positive cycle then state (A, π) cannot become envy-freeable.

Proof. Assume a state (A, π) and its induced Envy-graph. W.l.o.g assume there is a positive cycle in the Envy-graph of size 2, say between agents i, j. What holds then by construction of the Envy-graph is $v_i(A_j) - v_i(A_i) + v_j(A_i) - v_j(A_j) > 0$ call this equation (1). From the LP now, we have the following two constraints:

$$\pi_j - \pi_i \geq v_i(A_j) - v_i(A_i)$$

$$\pi_i - \pi_j \geq v_j(A_i) - v_j(A_j)$$

adding these inequalities we get

$$0 \geq v_i(A_j) - v_i(A_i) + v_j(A_i) - v_j(A_j)$$

which is a contradiction since the right term is strictly positive by hypothesis. This proves that the non existence of positive cycles is a necessary condition the Envy-graph must have in order for his state to become EF with payments. □

Remember that the reason for allowing monetary side payments is exclusively to help us deal with non envy-freeness. We already know that there exist instances that do not have any EF allocation at all (remember the example with two agents and one good with positive valuations). Below we give an important lemma on envy-freeness and SW_{util}.

Lemma 5. Given an allocation A, the following statement holds: If A is envy-freeable then A maximizes the utilitarian social welfare across all reassignments of bundles to agents, namely for all permutations σ of N, $\sum_{i \in N} v_i(A_i) \geq \sum_{i \in N} v_i(A_{\sigma(i)})$

Proof. Suppose A is envy-freeable. Then there exist payment balances s.t $\forall i, j \in N$, $v_i(A_i) + \pi_i \geq v_i(A_j) + \pi_j$, that is $\pi_i - \pi_j \geq v_i(A_j) - v_i(A_i)$. Now consider any permutation $\sigma(i)$ of N and sum the above relations. We get that $\sum_{i \in N}(\pi_i - \pi_{\sigma(i)}) \geq \sum_{i \in N}(v_i(A_{\sigma(i)}) - v_i(A_i)) \Rightarrow \sum_{i \in N} v_i(A_i) \geq \sum_{i \in N} v_i(A_\sigma(i))$. □

Keep in mind again that the above results hold in the monetary side payments model. In the subsidy model, which we will present in the last chapter, even stronger relations hold. Actually in this model we can prove that using subsidy an EF allocation always exist. Since equivalence of these models has not been proved yet the above results are what we can prove right now.

2.8.2 Converges to EEF

Returning to the main question whether there exists a distributed procedure to guarantee converges to an EEF state, we show in this section that again there exists a payment scheme that when imposed guarantees converges to EEF states. In a similar thought process like proportionality we know that IR deals lead the system to efficient allocations. Now our goal is to find proper payments such that this efficient state is EF too.

Notice that the efficient allocation is the allocation with the highest SW_{util}, which means that any reallocation of bundles only decreases the utilitarian social welfare. Assume an efficient allocation A^*. Since this is the efficient allocation we know that if we give agent i agent j's bundle, call this allocation B, the systems utilitarian welfare does not increase. It holds that:

$$SW_{util}(A^*) \geq SW_{util}(B) \Rightarrow v_i(A_i^*) + v_j(A_j^*) \geq v_i(A_i^* \cup A_j^*) \Rightarrow v_i(A_i^*) + v_j(A_j^*) \geq v_i(A_i^*) + v_i(A_j^*)$$

. Notice that $v_i(A_i^* \cup A_j^*) \geq v_i(A_i^*) + v_i(A_j^*)$, holds if and only if the valuation functions of the agents are supermodular. Eventually we get the relation that holds always only for the efficient allocation under supermodular valuations, namely that:

$$v_i(A_i^*) \geq v_i(A_j^*) \tag{2.3}$$

We keep inequality (2.3) in mind for now. The thought process is again the same as in the proportional case. Since the system starts from a random initial allocation and converges to the efficient allocation in a distributed way, the initial allocation could be the efficient one. So we need to find an initial payment to impose on the agents so as any initial allocation is EF. Below we give the proof that the efficient allocation accompanied with equitability payments is EF.

Lemma. *Efficient allocation accompanied by equitability payments is EF.*

Proof. Consider payments $\pi_i^* = v_i(A) - SW_{util}(A)/n$. We refer to these payments as initial equitability payments. It is easy to check that π^* is a payment function ($\sum_{i \in N}[v_i(A) - SW_{util}(A)/n] = SW_{util}(A) - SW_{util}(A) = 0$). Can the efficient allocation become EF with these payments? Assume an efficient allocation A^* and the state (A^*, π^*). For the state to be EF it must hold that $u_i(A_i^*, \pi_i^*) \geq u_i(A_j^*, \pi_j^*)$. This means that $v_i(A_i^*) - \pi_i^* \geq v_i(A_j^*) - \pi_j^*$, $\forall\ i, j \in N$. Since $\pi_i^* = v_i(A_i^*) - SW_{util}(A^*)/n$ we get the relation $SW_{util}(A^*)/n \geq v_i(A_j^*) - v_j(A_j^*) + SW_{util}(A^*)/n \iff v_j(A_j^*) \geq v_i(A_j^*)$, which holds from inequality (2.3) thus the efficient allocation with initial equitability payments π^* is EF. □

Now that we know the payment balance that gives an EEF we just have to determine which payment function to use so as to guarantee that whenever the system reaches the efficient allocation the total payment will be π^*.

Again we do not know when the system will reach the efficient allocation so we want at any reached allocation the payments to be of this form. Assume w.l.o.g that the system is at state (A, π) and an IR deal exists to jump to state (B, π'). Then for the payments made in this transition it holds that $p_i = \pi'_i - \pi_i$, and we want both π' and π to have the form of the initial equitability payments. So the payment we want is

$$p_i = \pi'_i - \pi_i = [v_i(A_i) - v_i(B_i)] - [SW_{util}(A) - SW_{util}(B)]/n$$

Which of course is the GUPF. Finally the payment scheme we must impose on the system is the initial equitability payments as initial payments and the GUPF as the payment function. That way, using IR deals, the system will always converge to an efficient allocation where the payment balance makes this allocation EF. A more detailed presentation can be found in [12].

Theorem 12 (Chevaleyre et al [12]). *Under supermodular valuation functions and the payment scheme consisting of the initial equitability payments as initial payments and the GUPF as payment function, the system using any IR deal sequence will always converge to an EEF state.*

Can this theorem hold for any kind of valuation function? In other words, does a general class of valuation functions still guarantee converges to EEF? If we keep the same payment scheme the answer is no and is stated in the next theorem

Theorem 13 (Chevaleyre et al [12]). ***Maximality of supermodularity:*** *No class F of valuation functions that strictly includes the class of supermodular functions can guarantee the following property: if all valuation functions are drawn from F and initial equitability payments have been made, then any sequence of IR deals using the GUPF will eventually result in an EEF state.*

Proof. Assume the payment scheme consisting of initial equitability payment + GUPF. Assume also an arbitrary allocation say B. For state (B, π) to be EF it must hold that $u_i(B_i, \pi_i) \geq u_i(B_j, \pi_j)$ which holds only if $v_j(B_j) \geq v_i(B_j)$, call this relation $(*)$. To prove the theorem we will construct a counterexample consisting of two agents and two goods where one agent has subadditive valuation function and the other has supermodular.

Assume $M = \{g_1, g_2\}$ and $N = \{1, 2\}$ with the following valuation: $v_1(\{g_1, g_2\}) = v_1(g_1) + v_1(g_2) - d$, $v_2(g_1) = v_1(g_1) - d/2$, $v_2(g_2) = v_1(g_2) - d/3$

It is easy to check that allocation $A = (g_1, g_2)$ is the efficient with $SW_{util}(A) = v_1(g_1) + v_2(g_2) = v_1(g_1) + v_1(g_2) - d/3$. But in A for agent 2 we have that $v_2(g_2) = v_1(g_2) - d/3 < v_1(g_2)$ which is a contradiction to $(*)$ □

2.9 Fair division on social networks

Apart from the lack of central authority to decide how to allocate the goods, the study of distributed fair allocations concerns real life negotiations where the agents are spatially distributed meaning that there are agents that do not now about the existence of others and the bundles they posses. This is a more realistic concept because in real life, agents are acting and negotiating in a local environment.

Real life social networks like facebook or twitter can be used as applications to this framework of study. In such networks the agents are only interested to the goods other agents have (other agents might be the friends in facebook, followers in twitter). Note that if someone cannot access the profile of a person in facebook he does not bother about the goods he posses. Moreover agents may not be aware of goods being held by agents outside their scope of visibility. To model a negotiation topology of agents we will use a graph which we will refer to as a topology graph or a social network. Again the general setup is that of a set of M goods and N agents and an undirected graph to model the network topology.

Definition (Negotiation topology). *A negotiation topology (or social network) is an undirected graph $G = (N, E)$ where N is the set of nodes that denotes the agents and E the set of edges, is a binary relation on the set of agents N where a pair $(i, j) \in E$ if and only if agent i can see agent j and vice versa.*

From the above definition it is clear that the notion of a deal must change in order to adapt to this definition. Since the topology defines a local interaction between agents, the deals must inherit this local notion too. In the classic distributed setup, notice that all agents have the same information on the available goods and who posses each good. It is important in the social network setup to ensure this same equality in information.

Consider a topology of agents where only one agent is aware of the bundles of all agents and the others are not. It is possible for this agent to conduct deals since he has full information that favour him while the rest of the systems efficiency is very low. We do not want to favour more any of the agents. Of course we cannot fully control this since the initial allocation is random but what we impose on the system is that any deal that is being made concerns a set of agents who have the same information for each other, namely agents that form a clique.

Definition (**Clique-deals**). *A deal* $\delta = (A, A')$ *is called a clique-deal if the set of involved agents* A^{δ} *is a clique of the negotiation topology* G.

Again we want to reach allocations that have maximum SW_{util}. We do not know yet if this is possible in this setup, but we know that IR deals are necessary for this.

2.9.1 Proportionality in social networks

The way to define proportionality on negotiation topologies though is not straightforward. In proportionality the agents have full information about every good and where it is allocated. If we define proportionality in the same classic way, it is possible that a proportional state will be never reached. The reason is that due to the structual constraint of the social network and the clique deals, the reached allocation, might be of very low efficiency due to low efficiency of some agents.

Another approach is to define proportionality based on goods the agent can only see which is again questionable, since the agents must forget about the goods they have seen so far. It is not yet straightforward how to define proportionality meaningfully, so as to find ways for the system to converge to proportional states.

2.9.2 Envy-freeness in Social Networks

Unlike proportionality, defining envy-freeness is more straightforward. An agent sees only a subset of the full set of agents, so he can only envy those agents. The definition of an envy-free state follows naturally.

Definition (**GEF, graph envy-free state**). *A state* (A, π) *is called graph envy-free* (GEF)*, with respect to the graph* $G = (N, E)$*, if and only if* $u_i(A_i, \pi_i) \geq u_i(A_j, \pi_j)$, $\forall\ (i, j) \in E$.

It is possible by this definition that for an agent i, a bundle of higher valuation exists but the agent cannot see the agent who possesses it so he is not envious of him. Also notice that this fact discourage our quest to reach allocations with maximum utilitarian welfare. Below we give an example of this claim.

Example. Assume an instance of three agents and only one good say g. Assume also the valuation of the agents as follows: $v_1(g) = 1, v_2(g) = 0, v_3(g) = 100$. Consider the initial random allocation that gives agent 1 good g and the negotiation topology as follows : $G = (\{1, 2, 3\}, \{12, 23\})$. Agent

2 sees both agents but he is not interested in g. Agent 1 possesses the good initially but agent 3 wants it more. Check that this initial state is GEF, since for agent 1 we get $u_1(A_1, \pi_1) = 1 > u_1(A_2, \pi_2) = 0$, for agent 2 the relation is obvious and for agent 3 $u_3(A_3, \pi_3) = 0 \geq u_3(A_2, \pi_2) = 0$. Note that we checked envy-freeness on every edge for both adjacent nodes. The SW_{util} of this allocation though is 1.

If we gave good g to agent 3 instead, then the system would have $SW_{util} = 100$. We see that the allocation is GEF but not efficient. Moreover this allocation can never become efficient since agent 2 has no interest in good g thus agent 3 could never conduct a deal to receive it. The reason is that agents only negotiate IR clique-deals and the only cliques in the topology are 12 and 23 where no deal on 12 is IR. Of course the situation would be different if agent 2 had the good initially.

2.9.3 Clique-wise efficiency

From the previous example we know that there exist negotiation topologies where the efficient allocation cannot be reached. So we know that there are instances with no hope to reach the optimal solution. In this subsection we present another approach to efficient allocations and prove whether this is actually a good approximation. First of all we need a new way to compare two different allocations.

Definition (Clique-variants). *Let A be an allocation. Another allocation B is called clique-variant of A if and only if there exists a clique $C \subseteq N$ such that $\bigcup_{i \in C} A_i = \bigcup_{i \in C} B_i$ and $A_i = B_i$, $\forall\ i \notin C$.*

Notice that A and B are clique-variants of each other if and only if $\delta(A, B)$ is a clique deal. Also observe that the clique-variant relation is not transitive. What we can do now, in terms of efficiency, is to compare clique-variants. We are interested in clique-variants of an allocation if they have higher utilitarian welfare until no other clique-variant with higher SW_{util} exists. These allocations are called clique-wise efficient.

Definition (Clique-wise efficiency). *An allocation A is called clique-wise efficient if and only if for any clique-variant B of A holds, $SW_{util}(A) \geq SW_{util}(B)$*

It is obvious though that there might exist more than one clique-wise efficient allocations and some of them might be far from the efficient allocation. We can prove that the clique-wise efficiency can be arbitrarily bad compared to the efficient allocation.

Proposition 6. If the negotiation topology is not complete then the utilitarian social welfare of a clique-wise efficient allocation can be arbitrarily far from the efficient allocation.

Proof. Assume an instance of n agents and two goods g_1, g_2 with $v(\{g_1, g_2\}) = 1$ for every agent and any other valuation is zero. Consider a negotiation topology that is not complete and w.l.o.g assume that no edge connects agents 1 and 2. Now take allocation A where agent 1 holds good g_1 and agent 2 holds good g_2 and $SW_{util}(A) = 0$. Allocation A is clique-wise efficient, since no adjacent edge to agents 1 and 2 exists there is no clique-deal to pass good g_2 from agent 2 to agent 1. Thus no clique-variant of A exists with higher SW_{util}. The efficient allocation of the system though is achieved when either agent 1 or agent 2 hold both goods, call this allocation B where $SW_{util}(B) = 1$. Now we can see that the ratio $\frac{SW_{util}(B)}{SW_{util}(A)}$ cannot be bounded by a constant thus it can be arbitrarily away from the optimum. □

So the structural constraints of the clique-deals and the negotiation topology makes the efficient allocation hard to approximate. It is still questionable whether we can relativise the standard notion of efficiency with respect to a negotiation topology. In complete negotiation topologies we know that IR deals eventually lead the system to the efficient allocation. We are about to prove some results in negotiation topologies that are not necessarily complete, concerning converges to an GEF state.

The goal is again the same, we want to find out whether it is possible for the agents, using clique-wise IR, deals to always converge to a clique-wise efficient allocation and if there is a specific payment scheme we can impose on them to guarantee converges to a GEF. First we figure out that IR clique deals are sufficient to lead the system to a clique-wise efficient allocation.

Theorem 14 (Chevaleyre et al [12])**.** *Any sequence of IR clique deals will eventually result in a clique-wise efficient allocation.*

Proof. First there cannot be an infinite sequence of IR clique deals since every deal increases the SW_{util} and the total number of allocations is finite. Assume the negotiations terminate in an allocation A and also assume A is not a clique-wise efficient allocation. That means a clique variant of A does exist that has higher utilitarian welfare. But if such an allocation exists then there must exist an IR clique deal, (IR since there is increase in SW_{util} and clique deal since the allocation is a clique variant of A). But this is a contradiction since in the hypothesis we assumed that the negotiations have terminated. □

We already proved that under supermodular valuations the payment scheme consisting of initial equitability payments and GUPF suffices to always lead the system to an EEF state. The next theorem proves the same for GEF states in social networks.

Theorem 15 (Chevaleyre et al [12])**.** *If all valuations are supermodular and if initial equitability payments have been made, then any sequence of IR clique-deals using the GUPF will eventually result in a GEF state.*

Proof. Remember that initial equitability payments combined with GUPF always ensures at any state (A, π) a payment balance $\pi_i = v_i(A_i) - SW_{util}(A)/n$ for every agent. Again if the system uses only IR clique deals the negotiations will eventually terminate and will terminate in a clique-wise efficient allocation as we proved in theorem 12.

Assume this clique-wise efficient allocation is A^*. We know that the payment balance at this allocation is $\pi^* = v_i(A^*) - SW_{util}(A^*)/n$. We show now that state (A^*, π^*) must be GEF. Consider two agents i, j. If agent i cannot see agent j then we are done. If they see each other, we use the same argument as in theorem 12. If we give both bundles A_i, A_j to agent i, then that does not increase the SW_{util} because if it did then that would be an IR clique deal leading to a higher utility allocation which contradicts the fact that state (A^*, π^*) is a clique-wise efficient state. Hence using exactly the same argument as in theorem 12 we prove that this state is GEF. □

2.10 Examples

In this section we are giving examples of distributed fair division using the Knaster scheme and the initial equitability payment and GUPF scheme.

Example 8. Consider the following instance consisting of three agents and four goods.

Agents	g_1	g_2	g_3	g_4
1	10	(1)	1	2
2	5	2	(1)	(3)
3	(5)	5	1	1

Assume that the system starts from the initial allocation $A_0 = (g_2, \{g_3, g_4\}, g_1)$. We will use the notation $v_i(A)$ to denote the valuation of agent i for the bundle he possesses in allocation A. For the agents holds: $v_1(A_0) = 1, v_2(A_0) = 4, v_3(A_0) = 5$ and $SW_{util}(A_0) = 10$. Next we compute the initial payments according to the Knaster scheme.

$ex_1(A_0) = v_1(A_0) - v_1(M)/3 = 1 - 14/3 = -11/3$

$ex_2(A_0) = v_2(A_0) - v_2(M)/3 = 4 - 11/3 = 1/3$

$ex_3(A_0) = v_3(A_0) - v_3(M)/3 = 5 - 4 = 1$

The total excess of allocation A_0 is : $EX(A_0) = -7/3$. The initial payments imposed by the Knaster scheme are:

$\pi_1(A_0) = ex_1(A_0) - EX(A_0)/3 = -26/9$

$\pi_2(A_0) = ex_2(A_0) - EX(A_0)/3 = 10/9$

$\pi_3(A_0) = ex_3(A_0) - EX(A_0)/3 = 16/9$

So the utilities of all 3 agents are:

$u_1(A_0, \pi_1) = 1 + 26/9 = 35/9$

$u_2(A_0, \pi_2) = 4 - 10/9 = 26/9$

$u_3(A_0, \pi_3) = 5 - 16/9 = 29/9$

Allocation A_0 is not proportional yet. Check that for agent 1 $u_1(A_0, \pi_1) = 35/9 < v_1(M)/3 = 14/3$.

Consider now that the system jumps to allocation $A_1 = (\{g_1, g_4\}, g_2, g_1)$.

Agents	g_1	g_2	g_3	g_4
1	(10)	1	1	(2)
2	5	(2)	1	3
3	(5)	5	1	1

Since $SW_{util}(A_1) = 12 + 2 + 1 = 15$, A_1 has higher utilitarian social welfare than A_0 and we know that an IR deal does exist to allow the system this transition. Since we are using the Knaster scheme we use the GUPF to determine each agents payments. Remember the formula of the GUPF is: $p_i = v_i(A_1) - v_1(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/n$ so for the three agents the payments are as follows:

$p_1 = v_1(A_1) - v_1(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/n = 12 - 1 - [15 - 10]/3 = 28/3$

$p_2 = v_2(A_1) - v_2(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/n = 2 - 4 - [15 - 10]/3 = -11/3$

$p_3 = v_3(A_1) - v_3(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/n = 2 - 4 - [15 - 10]/3 = -7/3$

We compute now the new excesses for allocation A_1.

$ex_1(A_1) = v_1(A_1) - v_1(M)/3 = 12 - 14/3 = 22/3$

$ex_2(A_1) = v_2(A_1) - v_2(M)/3 = 2 - 11/3 = -5/3$

$ex_3(A_1) = v_3(A_1) - v_3(M)/3 = 1 - 12/3 = -3$

The total excess of allocation A_1 is : $EX(A_1) = 8/3$

The payment balance in allocation A_1 is : $\pi_1 = ex_1(A_1) - EX(A_1)/3 = 22/3 - 8/9 = 58/9$

$\pi_2 = ex_2(A_1) - EX(A_1)/3 = -5/3 - 8/9 = -23/9$

$\pi_3 = ex_3(A_1) - EX(A_1)/3 = -3 - 8/9 = -35/9$

And the utilities in state (A_1, π) are :

$u_1(A_1, \pi_1) = 12 - 58/9 = 50/9$

$u_2(A_1, \pi_2) = 2 + 23/9 = 41/9$

$u_3(A_1, \pi_3) = 1 + 35/9 = 44/9$

We can check that allocation A_1 is proportional: $u_1(A_1, \pi) = 50/9 > v_1(M)/3 = 14/3$

$u_2(A_1, \pi) = 41/9 > v_2(M)/3 = 11/3$

$u_3(A_1, \pi) = 44/9 > v_3(M)/3 = 12/3$

Although the system has reached a proportional allocation this allocation is not efficient. If we let the system negotiate again they will eventually reach allocation $(\{g_1, g_4\}, g_3, g_2)$ with $SWutil = 18$ which is efficient. If we compute the new payment balances and utilities we can see that this allocation is both proportional and efficient.

Example 9. Consider again the same instance.

Agents	g_1	g_2	g_3	g_4
1	10	(1)	1	2
2	5	2	(1)	(3)
3	(5)	5	1	1

Assume again the same initial allocation $A_0 = (g_2, \{g_3, g_4\}, g_1)$, with $SW_{util}(A_0) = 10$. We compute the initial equitability payments.

$\pi_1 = v_1(A_0) - SW_{util}(A_0)/3 = 1 - 10/3 = -7/3$

$\pi_2 = v_1(A_0) - SW_{util}(A_0)/3 = 4 - 10/3 = 2/3$

$\pi_3 = v_1(A_0) - SW_{util}(A_0)/3 = 5 - 10/3 = 5/3$

The utilities are as follows:

$u_1(A_0, \pi_1) = v_1(A_0) - \pi_1 = 1 - (-7/3) = 10/3$

$u_2(A_0, \pi_2) = v_2(A_0) - \pi_2 = 4 - 2/3 = 10/3$

$u_3(A_0, \pi_3) = v_3(A_0) - \pi_3 = 5 - 5/3 = 10/3$

We can see that allocation A_0 is not EF since agent 2 is envious of agent 1 : $u_2(A_0(1), \pi_1) = 2 + 7/3 = 13/3 > u_2(A_0(2), \pi_2) = 10/3$.

Consider now that the system jumps to allocation $A_1 = (\{g_1, g_4\}, g_3, g_2)$,

Agents	g_1	g_2	g_3	g_4
1	(10)	1	1	(2)
2	5	2	(1)	3
3	5	(5)	1	1

where $SW_{util}(A_1) = 18$. The payments for each agent are computed using the GUPF : $p_1 = v_1(A_1) - v_1(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/3 = 12 - 1 - [18 - 10]/3 = 11 - 8/3 = 25/3$

$p_2 = v_2(A_1) - v_2(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/3 = -3 - 8/3 = -17/3$

$p_3 = v_3(A_1) - v_3(A_0) - [SW_{util}(A_1) - SW_{util}(A_0)]/3 = -8/3$

The payment balance at the new allocation is:

$\pi_1 = v_1(A_1) - SW_{util}(A_1)/3 = 12 - 18/3 = 6$

$\pi_2 = v_2(A_1) - SW_{util}(A_1)/3 = 1 - 18/3 = -5$

$\pi_3 = v_3(A_1) - SW_{util}(A_1)/3 = 5 - 18/3 = -1$

And the utilities of the agents at A_1 are:

$u_1(A_1, \pi_1) = v_1(A_1) - \pi_1 = 12 - 18/3 = 6$

$u_2(A_1, \pi_2) = v_2(A_1) - \pi_2 = 1 - (-5) = 6$

$u_3(A_1, \pi_3) = v_3(A_1) - \pi_3 = 5 - (-1) = 6$

It is easy to check that this allocation is EF and it is efficient too.

Example 10. Now we give an example to illustrate the LP, the envy-graph and the positive cycles. Consider the following instance:

Agents	g_1	g_2	g_3
1	(1)	1	2
2	3	2	(1)
3	2	(2)	4

Assume allocation $A = (g_1, g_3, g_2)$. The LP for this allocation is as follows:

$$\begin{aligned}
&min \quad \{\emptyset\} \\
&\text{subject to:} \\
&\pi_1 - \pi_2 \geq v_2(A_1) - v_2(A_2) \\
&\pi_1 - \pi_3 \geq v_3(A_1) - v_3(A_3) \\
&\pi_2 - \pi_1 \geq v_1(A_2) - v_1(A_1) \\
&\pi_2 - \pi_3 \geq v_3(A_2) - v_3(A_3) \\
&\pi_3 - \pi_1 \geq v_1(A_3) - v_1(A_1) \\
&\pi_3 - \pi_2 \geq v_2(A_3) - v_2(A_2) \\
&\sum_{i=1}^{n} \pi_i = 0 \\
&\forall i, j \in N, i \neq j
\end{aligned}$$

Where for the inequality constraints holds:

$$\begin{aligned}
&\pi_1 - \pi_2 \geq 2 \\
&\pi_1 - \pi_3 \geq 0 \\
&\pi_2 - \pi_1 \geq 1 \\
&\pi_2 - \pi_3 \geq 2 \\
&\pi_3 - \pi_1 \geq 0 \\
&\pi_3 - \pi_2 \geq 1 \\
&\pi_1 + \pi_2 + \pi_3 = 0
\end{aligned}$$

Remember that the envy-graph is a complete directed graph whose nodes are the agents and an edge from agent i to agent j has the assigned value $v_i(A_j) - v_i(A_i)$. We have proved that an allocation can become EF with payments only if the envy-graph has no positive cycles. In this example we can see that the envy-graph of allocation A has a positive cycle, check cycle 1221. For this cycle holds $v_1(A_2) - v_1(A_1) + v_2(A_1) - v_2(A_2) = 2 - 1 + 3 - 1 = 3$ thus this allocation cannot become EF with payments.

2.11 Negotiations without payments

There are also negotiations without money. Such negotiations do not have as an objective converge to EEF states, but to maximize another social welfare like the egalitarian or they have as an objective to reach a Pareto optimal allocation. In this section we briefly present some theoretic results on negotiations without money. The majority of ideas and concepts are presented by Endriss et al in [15]

2.11.1 Ensuring Pareto optimal outcomes

Obviously since our objective has changed, we need different kind of deals to ensure the desired outcome. We know that IR deals always increase each agents utility. From the definition of Pareto dominance we know that at least one agent's utility must be strictly higher in the new allocation, so as to be preferred from the previous one. The deals we allow the system to make must ensure this property. The deals we impose on the system are called cooperatively rational.

Definition (Cooperatively rational deals). *A deal $\delta = (A, A')$ is called cooperatively rational if and only if $u_i(A_i) \leq u_i(A'_i)$, $\forall\ i, j \in N$ and the inequality is strict for at least one agent, namely, $\exists\ i \in N$ such that $u_i(A_i) < u_i(A'_i)$.*

Notice that cooperative rational deals like IR deals also strictly increase the SW_{util} of the system, but unlike IR deals an increase in SW_{util} does not always imply the existence of a cooperative rational deal. We only know that a cooperatively rational deal increases the SW_{util} but not vice versa. Cooperatively rational deals are sufficient to lead the system to a Pareto optimal allocation.

Theorem 16 (Endriss et al [15]). *Any sequence of cooperatively rational deals will eventually result in a Pareto optimal allocation.*

Proof. Every cooperatively rational deal strictly increases the SW_{util}. Combined with the fact that there exist only a finite number of allocations, this implies that any negotiation process will eventually terminate. Assume for the sake of contradiction, that negotiations terminate in an allocation A which is not Pareto optimal. That means there exists another allocation say A' with $SW_{util}(A) < SW_{util}(A')$ and $u_i(A_i) \leq u_i(A'_i)$, $\forall\ i \in N$. If we had for every agent that $u_i(A_i) = u_i(A'_i)$ then we would had also that $SW_{util}(A) = SW_{util}(A')$, but since $SW_{util}(A') > SW_{util}(A)$ there must be an agent say j for whom it holds that $u_j(A_j) < u_j(A'_j)$. Then the deal $\delta = (A, A')$ would be a cooperatively rational deal which contradicts the assumption that the negotiations have terminated. □

2.11.2 Ensuring maximum egalitarian welfare outcomes

In this framework of study our objective is to find the allocation with the highest SW_{egal}. Again the structure of the allowed deals must change to adapt to our quest target. Remember that the egalitarian welfare has to do with the minimum utility of an agent in the system. What we want is eventually to find the allocation that has the maximal SW_{egal}.

It is straightforward that cooperatively rational deals will not do the work in this framework of study. What is the main idea behind the egalitarian social welfare is that an allocation would be closer to the optimal allocation if its minimum utility agent has higher utility than the minimum utility agent in the previous allocation. So the deals the agents agree on, must have the property to increase the minimum utility. To formalize this idea we use a class of deals called equitable deals.

Definition (Equitable deals). *A deal $\delta = (A, A')$ is called equitable if and only if it satisfies*

$$min\{u_i(A)|i \in A^\delta\} < min\{u_i(A')|i \in A^\delta\}$$

Observe that the above deal structure has an effect only on the agents that participate in the deal. It is easy to see now that whenever the system jumps in a new allocation with higher egalitarian welfare, an equitable deal exists to make this transition.

Lemma 6. If A and A' are allocations with $SW_{egal}(A) < SW_{egal}(A')$ then $\delta(A, A')$ must be an equitable deal.

Proof. Assume allocations A, A' with $SW_{egal}(A) < SW_{egal}(A')$ and the deal $\delta(A, A')$. We will prove that deal δ is an equitable deal. First observe that the agent with the minimum utility must participate in δ, otherwise the egalitarian welfare of A' would be the same as A. So what holds is that: $\min\{u_i(A)|i \in A^\delta\} = SW_{egal}(A)$, (1). For all the agents participating in the deal it holds $u_i(A') \geq SW_{egal}(A')$ and from the assumption we have that $SW_{egal}(A) < SW_{egal}(A')$ which along with (1) implies that: $\min\{u_i(A)|i \in A^\delta\} = SW_{egal}(A) < SW_{egal}(A') \leq min\{u_i(A')|i \in A^\delta\}$. Thus $\min\{u_i(A)|i \in A^\delta\} < \min\{u_i(A')|i \in A^\delta\}$ which proves that deal δ is indeed an equitable deal. □

But still we do not get a solid result on whether equitable deals are sufficient to reach the optimal allocation. The other direction of the lemma does not hold. It is not the fact that every equitable deal increases the egalitarian welfare (for example if the participants are the agents with the two higher utilities). What we can prove though about equitable deals is that they imply leximin-rise.

Lemma 7. If $\delta = (A, A')$ is an equitable deal then $A <_{lex} A'$.

Proof. Let $\delta = (A, A')$ be an equitable deal an let $a = \min\{u_i(A)|i \in A^\delta\}$. We are about to prove that $A <_{lex} A'$. Consider the ordered utility vector of allocation A say u_A. Since δ is a deal, the utilities of the agents that change are only those of the participants in the deal and since a is the minimum utility of the participants, we know that in vector u_A all values preceding a are the same. W.l.o.g let's assume that value a belongs to the k-th utility in the ordered utility vector of allocation A. Since the deal is equitable from its definition we know that this minimum value a is increased in the new allocation A'.

Now two possible scenarios exist. The first is that the value a increases but is still the k-th value in the ordered utility vector of allocation A', which implies that $u_A <_{lex} u_{A'}$. The second is that a increases again and another agent's utility is now in the k-th order, say this utility is b. We know that before b all other utilities are the same since those agents do not participate in the deal. So it suffice to show that utility b is greater than utility a. But this is obvious since in u_A utility b must be in higher order than a thus the k-th component of $u_{A'}$ is greater than the k-th component of u_A meaning that $u_A <_{lex} u_{A'}$ and that proves our claim. □

Once again not any leximin-rise implies the existence of an equitable deal. Consider deals that leave the utility of the weakest agent the same. Despite the fact that we cannot find an equivalence relation among the egalitarian welfare and equitable deals we have the tools to prove the converges result we want.

Theorem 17 (Endriss [15])**.** *Any sequence of equitable deals eventually results in an allocation with maximal egalitarian welfare.*

Proof. By Lemma 7 we know that every equitable deal results in a strict rise in the leximin ordering. Hence as there is only a finite number of allocations, the negotiations will terminate after a finite number of deals. Suppose that negotiations have terminated and no other equitable deal exists. Let A be the terminal allocation. Assume that A is not the allocation with the maximal egalitarian welfare. Then another allocation say A' exists that has higher SW_{egal}. Then from Lemma 6 there must exist an equitable deal $\delta = (A, A')$ to make the transition from A to A', but this is a contradiction since we assumed that the negotiations have terminated. Thus allocation A is the allocation with the maximal egalitarian welfare. □

We must be very careful in understanding what we proved. Eventually the maximal egalitarian welfare allocation will be reached, that does not mean though that no other equitable deal exists. Keep in mind that equitable deals result always in a leximin rise so it is possible that a rise not in the weakest agent's utility but some other agent's utility does occur. Below we give an illustrative example.

Example. Consider the following instance:

Agents	g_1	g_2	g_3	$\{g_1, g_2, g_3\}$
1	5	0	0	5
2	ε	7	7.5	7.5
3	ε	9	8.5	9.5

Assume the valuation for the bundles not shown in the table is zero. Assume allocation A where agent 3 holds all the goods initially. The utility vector of this allocation is $u_A = \langle 0, 0, 9.5 \rangle$. If agent 3 gives agent 1 good g_1 and agent 2 g_2 then this is an equitable deal and increases the egalitarian welfare since the new utility vector is $\langle 5, 7, 8.5 \rangle$.

Agents	g_1	g_2	g_3	$\{g_1, g_2, g_3\}$
1	(5)	0	0	5
2	ε	(7)	7.5	7.5
3	ε	9	(8.5)	9.5

This allocation is also the allocation with the maximal egalitarian welfare. Notice though that another equitable deal does exist, agent 3 and agent 2 can swap goods g_2 and g_3 thus achieving a utility vector $\langle 5, 7.5, 9 \rangle$.

Agents	g_1	g_2	g_3	$\{g_1, g_2, g_3\}$
1	(5)	0	0	5
2	ε	7	7.5	(7.5)
3	ε	(9)	8.5	9.5

The new allocation is higher than the previous with respect to the lexicographic order but its egalitarian welfare remains the same.

2.12 Summary

In this chapter we discussed distributed fair division. The general setup was the same as in centralized fair division with the exception that the agents negotiate between them deals which help them improve their utilities. In these negotiations money are used to help the system to reach an efficient and fair allocations. We defined the utilities of the agents based on the monetary side payment model, which are the objectives upon which the agents form the deals to conduct.

We discussed all available deal types and structures and analysed each one separately. We proved via counterexamples that none of these types alone is sufficient to help the system converge to efficient allocations thus concluding to OCSM-deals that helps the system eventually converge to efficient allocations. Next we presented two payments schemes used in bibliography to guarantee converges to efficient proportional states, the Knaster payment scheme, and converges to EEF states, the initial equitability payments + GUPF scheme. We presented some theoretic results concerning distributed fair division on social networks where the agents form a graph and have information only for the goods their neighbours posses. We presented some efficiency and fairness results on this model, with the most important the payment scheme that leads to GEF sates.

A separate section composed of examples on the definitions and notions was presented to illustrate the ideas. Finally we presented briefly some results on negotiations without side payments where the objective is to maximize another social welfare and focused on converges to the maximal egalitarian welfare allocation and to Pareto optimal allocations.

Chapter 3

Computing envy-freeable allocations with limited subsidies

3.1 Abstract

Having presented some basic notions on fair division of indivisible goods in chapters 1 and 2, in this chapter we present our contributions to the field. We will present the results we proved in our research while working on this thesis, concerning fair division in the subsidy model.

We consider the natural optimization problem of computing allocations that are *envy-freeable* using the minimum amount of subsidies. As the problem is NP-hard, we focus on the design of approximation algorithms, see [33]. On the positive side, we present an algorithm which, for a constant number of agents, approximates the minimum amount of subsidies within any required accuracy, at the expense of a graceful increase in the running time. On the negative side, we show that, for a superconstant number of agents, the problem of minimizing subsidies for envy-freeness is not only hard to compute exactly but also, more importantly, hard to approximate.

Remember that, with indivisible items, envy-freeness is rarely a feasible goal. For example, no such allocation exists in the embarrassingly simple case with a single item and two agents with some value for it. Recently proposed relaxations of envy-freeness, see section 1.5.2, aim to serve as useful alternative fairness notions. As we already discussed in chapter 2, allocations might be complemented with payments (or subsidies) to the agents. Now, envy-freeness dictates that no agent prefers the allocation and payment of another agent to his. Envy-freeness is a feasible goal

now. However, it poses questions related to the sparing use of money.

We follow an optimization approach. We define and study the optimization problem SMEF (standing for Subsidy Minimization for Envy-Freeness). Given an allocation problem consisting of items and agents with valuations for the items, SMEF asks for an allocation that is envy-freeable using the minimum total amount of subsidies.

SMEF is NP-hard; this follows by the NP-hardness of deciding whether a given allocation problem has an envy-free allocation or not. Thus, we resort to approximation algorithms for SMEF. As multiplicative approximation guarantees are hopeless, our aim is to design algorithms that run in polynomial-time and compute an allocation that is envy-freeable with an amount of subsidies that does not exceed the minimum possible amount of subsidies (denoted χ) by much. In particular, we use the total valuation of all agents for all goods (denoted by $\text{sum}\, v$) as benchmark and seek allocations that are envy-freeable with an amount of at most $\chi + \rho \cdot \text{sum}\, v$ as subsidies. The goal for the approximation guarantee ρ of an algorithm is to be as small as possible.

We initiate the study of SMEF and present two results. On the positive side, we design an algorithm that achieves an arbitrary low approximation guarantee of $\epsilon > 0$. When applied to allocation instances with a constant number of agents, the algorithm uses dynamic programming and runs in time that is polynomial in the number of items and $1/\epsilon$. On the negative side, we show that, in general, SMEF is not only hard to solve exactly, but also hard to approximate within a small constant. Unlike the folklore reduction[1] for proving hardness of envy-freeness, our proof uses a novel approximation-preserving reduction. Besides separating the general case from that with constantly many agents, our negative result indicates that achieving good approximation guarantees will be a challenging goal.

3.2 Preliminaries

Below we present in detail the model we used to work. Although many ideas might have been presented already we present them again for completeness. We consider allocation instances with a set M of m items and a set N of n agents. Each agent $i \in N$ has a valuation function $v_i : M \to \mathbb{R}_{\geq 0}$ over the items. With some abuse of notation, we use $v_i(B)$ to denote the valuation of agent i for

[1]Notice that deciding whether an envy-free allocation exists for two agents with identical item valuations requires solving PARTITION a well known NP-hard problem.

the set (or *bundle*) of items B. Valuations are additive, i.e., $v_i(B) = \sum_{g \in B} v_i(g)$. . We use the abbreviations $\operatorname{sum} v = \sum_{i \in N} v_i(M)$ and $\max v = \max_{i \in N} v_i(M)$. We define the *social welfare* of an allocation $X = (X_1, ..., X_n)$ to be the utilitarian, $\mathrm{SW}(X, v) = \sum_{i \in N} v_i(X_i)$.

For an allocation $X = (X_1, ..., X_n)$ in an instance with agent valuations v, the *envy graph* $\mathrm{EG}(X, v)$ is an edge-weighted complete directed graph that has a node for each agent and the weight of the directed edge (i, j) represents the "envy" of agent i for agent j. Using $G = \mathrm{EG}(X, v)$ and $\mathrm{wgt}_G(i, j)$ for the weight of the directed edge from node i to node j in the envy graph $\mathrm{EG}(X, v)$, we define $\mathrm{wgt}_G(i, j) = v_i(X_j) - v_i(X_i)$.

We consider *payments* (or *subsidies*) to the agents, represented by a payment vector $\pi = \langle \pi_1, ..., \pi_n \rangle$ with non-negative entries, i.e., $\pi_i \geq 0$ for every agent $i \in N$. The fact that the payments are non-negative decouples the monetary side payments model from the subsidy model. Below, we use the terms "payment" and "subsidy" interchangeably. Now, we say that the pair (X, π) of the allocation X and payment vector π is envy-free if $v_i(X_i) + \pi_i \geq v_i(X_j) + \pi_j$ for every pair of agents $i, j \in N$. Informally, this extended version of envy-freeness requires that no agent envies the bundle and the payment of any other agent compared to the bundle and payment he gets.

We say that an allocation X is *envy-freeable* if there is a payment vector π so that the pair (X, π) is envy-free. Even though the use of payments makes the goal of envy-freeness feasible, not all allocations are envy-freeable. The following theorem, due to Halpern and Shah [19], gives sufficient and necessary conditions that must hold for an allocation so as to be envy-freeable.

Theorem 18 (Halpern and Shah [19])**.** *The following statements are equivalent:*

1. *The allocation $X = (X_1, X_2, ..., X_n)$ is envy-freeable.*

2. *The allocation X maximizes social welfare among all redistributions of its bundles to the agents.*

3. *The envy graph $EG(X, v)$ contains no cycles of positive total weight.*

Proof. (1) $\Rightarrow$ (2): Suppose X is envy-freeable. Then there exists a payment vector π s.t $\forall i, j \in N$, $v_i(X_i) + \pi_i \geq v_i(X_j) + \pi_j$. That is $v_i(X_j) - v_i(X_i) \leq \pi_i - \pi_j$. Consider any permutation σ of $[n]$. Then $\sum_{i \in N}(v_i(X_{\sigma(i)}) - v_i(X_i)) \leq \sum_{i \in N}(\pi_i - \pi_{\sigma(i)}) = 0$.

(2) $\Rightarrow$ (3): Suppose condition (2) holds. Consider a cycle $C = (i_1, ...i_k)$ in $EG(X, v)$. Consider the corresponding permutation σ_C under which $\sigma(i_t) = i_t + 1$ for each $t \in [k-1]$, and $\sigma(i) = i$ for

all $i \notin C$. Then

$$\begin{aligned}
\text{wgt}(C) &= \sum_{t=1}^{k-1} wgt(i_t, i_t + 1) \\
&= \sum_{t=1}^{k-1} v_{i_t}(X_{i_{t+1}}) - v_{i_t}(X_{i_t}) \\
&= \sum_{t=1}^{k-1} (v_{i_t}(X_{i_{t+1}}) - v_{i_t}(X_{i_t})) + \sum_{i \notin C} (v_i(X_i) - v_i(X_i)) \\
&= \sum_{i \in N} (v_i(X_{\sigma(i)}) - v_i(X_i)) \leq 0.
\end{aligned}$$

(3) $\Rightarrow$ (1): Suppose $EG(X, v)$ has no positive cycles. Then $l(i)$, the maximum weight of any path starting from i in $EG(X, v)$ is well defined and finite. Let $\pi_i = l(i)$ for each $i \in N$. Note that $\pi_i \geq l(i, i) \geq \text{wgt}(i, i) = 0$ for each $i \in N$. Hence π is a valid payment vector. Also by definition of longest paths we have for all $i, j \in N$ $\pi_i = l(i) \geq l(j) + \text{wgt}(i, j) = \pi_j + v_i(X_j) - v_i(X_i)$. Hence $\pi_i - \pi_j \geq v_i(X_j) - v_i(X_i)$ thus X is envy-freeable. □

Detecting whether a given allocation X is envy-freeable can be done using the following linear program LP(X, v):

$$\begin{aligned}
\text{minimize} \quad & \sum_{i \in N} \pi_i \qquad (3.1) \\
\text{subject to:} \quad & \pi_i - \pi_j \geq v_i(X_j) - v_i(X_i), \forall i, j \in N \\
& \pi \geq 0
\end{aligned}$$

LP(X, v) aims to find a payment vector π so that the envy-freeness constraints between pairs of agents are satisfied. In addition, it minimizes the total amount of payments. As it is observed by Halpern and Shah [19], the payment π_i of agent i obtained in this way is equal to the maximum total weight in any simple path that originates from node i in the envy graph EG(X, v).

We study the optimization problem SMEF (standing for Subsidy Minimization for Envy-Freeness). Given an allocation instance, SMEF aims to compute an allocation that is envy-freeable with the minimum amount of subsidies. Since the problem of computing an envy-free allocation is NP-hard, SMEF is NP-hard as well.

We are interested in the design of approximation algorithms for SMEF. As algorithms with finite multiplicative approximation ratio are hopeless (since it is hard to decide whether the minimum amount of subsidies is zero or not), we seek polynomial-time algorithms that compute an allocation that is envy-freeable with subsidies $\chi + \rho \cdot \operatorname{sum} v$, with the approximation guarantee ρ being as low as possible.

As a warmup, consider the algorithm that allocates all items to the agent i^* that has maximum value for M and paying a subsidy of $v_{i^*}(M)$ to every other agent i. This algorithm is clearly polynomial-time. The allocation obtained is envy-freeable since no redistribution of the items (i.e., giving all items to another agent) results to a higher social welfare. And the particular payments are right: agent i^* is indifferent between the bundle M and the payment to any other agent, while the other agents are indifferent between the (equal) payments, and prefers their payment to getting the whole bundle M. It can be easily verified that the algorithm guarantees an amount of at most $\chi + (n-1)\max v \leq \chi + (n-1)\operatorname{sum} v$ as subsidies; this is the best guarantee of this form for this algorithm in the worst-case.

3.3 An approximation algorithm

We now present an algorithm that does much better. The algorithm exploits ideas that have led to poylynomial-time approximation schemes for combinatorial optimization problems like Knapsack (e.g., see [33]). It first discretize all valuations to multiples of a discretization parameter. In this way, the different discretized valuations an agent can have for bundles of items in the new instance is small. This allows to classify all allocations into a relatively small number of classes, each defined by specific discretized valuation levels of each agent for all bundles. Dynamic programming is used to decide the classes that are non-empty and to select a representative allocation from each class. The final allocation is selected among all representative allocations, possibly after redistributing the bundles so that social welfare (with respect to the original valuations) is maximized (in order to get envy-freeability). This requires a call to linear program (3.1) to compute the minimum amount of subsidies for each representative allocation.

The classification of allocations guarantees that the algorithm will consider a representative allocation from the class that also contains the optimal one (i.e., the allocation that is envy-freeable with the minimum amount of subsidies overall). Our analysis shows that the amount of subsidies for making the representative allocation envy-free is close to optimal. Polynomial running time for the case of a constant number of agents follows by setting the discretization parameter appropriately.

We now present our algorithm in detail. It uses an accuracy parameter $\epsilon > 0$ and initially decides the value of the discretization parameter δ as follows:

$$\delta = \frac{\epsilon \max v}{4mn^2}.$$

First, the algorithm implicitly discretizes all agent valuations by defining new valuations $\tilde{v}$ as follows: for an agent i with valuation $v_i(g)$ for item g, the discretized valuation $\tilde{v}_i(g)$ is equal to $\lfloor v_i(g)/\delta \rfloor \delta$.

The algorithm uses an arbitrary ordering of the items in M; let $M = \{g_1, g_2, ..., g_m\}$, where the item indices are those in this ordering. The algorithm builds a table $\mathbf{T}$ which classifies all possible allocations of subsets of M. Consider an (n^2+1)-dimensional tuple $\tau = (t, P_{ij}, 1 \leq i, j \leq n)$, where t is an integer from 1 to m and P_{ij} is an integer from 0 to $\lfloor \max v/\delta \rfloor$, for every pair of agents i and j. The entry $\mathbf{T}(\tau)$ of the table indicates whether an allocation $A^t = (A^t_1, A^t_2, ..., A^t_n)$ of the first t items $g_1, ..., g_t$ of M to the n agents, satisfying $\tilde{v}_i(A^t_j) = P_{ij}\delta$ for every pair of agents i and j, exists ($\mathbf{T}(\tau) = 1$) or not ($\mathbf{T}(\tau) = 0$).

The entries of $\mathbf{T}$ are computed using the following recursive relation:

- For a tuple $\tau = (t, P_{ij}, 1 \leq i, j \leq n)$ with $t = 1$, the algorithm sets $\mathbf{T}(\tau) = 1$ if there exists $k \in [n]$ such that, for every $i \in [n]$, $\tilde{v}_i(g_1) = P_{ik}\delta$ and $P_{ij} = 0$ for every $j \neq k$. Otherwise, the algorithm sets $\mathbf{T}(\tau) = 0$.

- For a tuple $\tau = (t, P_{ij}, 1 \leq i, j \leq n)$ with $t > 1$, the algorithm sets $\mathbf{T}(\tau) = 1$ if there exists $k \in [n]$ and tuple $\tau' = (t-1, P'_{ij}, 1 \leq i, j \leq n)$ such that, for every $i \in [n]$, $P_{ik} = P'_{ik} + \tilde{v}_i(g_t)/\delta$ and $P_{ij} = P'_{ij}$ for every $j \neq k$. Otherwise, the algorithm sets $\mathbf{T}(\tau) = 0$.

Essentially, each non-zero entry of $\mathbf{T}$ (e.g., $\mathbf{T}(\tau) = 1$) indicates a non-empty class $\mathcal{A}_\tau$ of (possibly partial, when the first argument of τ is an integer smaller than m) allocations. To compute a representative complete allocation $A_\tau \in \mathcal{A}_\tau$ among those implied by the non-zero entry corresponding to the tuple $(m, P^m_{ij}, 1 \leq i, j \leq n)$, the algorithm does the following for $t = m$ downto 2. Let $k \in [n]$ be such that $\mathbf{T}(\tau') = 1$ for a tuple $\tau' = (t-1, P^{t-1}_{ij}, 1 \leq i, j \leq n)$ with $P^{t-1}_{ik} = P^t_{ik} - \tilde{v}_i(g_t)/\delta$ and $P^{t-1}_{ij} = P^t_{ij}$ for every pair of agents i and $j \neq k$. The algorithm assigns item g_t to agent k and proceeds to considering the next item. The first item g_1 is assigned to agent k such that $\mathbf{T}(\tau') = 1$

for a tuple $\tau' = (1, P^1_{ij}, 1 \leq i, j \leq n)$ with $P^1_{ik} = \tilde{v}_i(g_1)/\delta$ and $P^1_{ij} = 0$ for every pair of agents i and $j \neq k$.

Next, the algorithm redistributes the bundles of each allocation A_τ that represents a non-empty class $\mathcal{A}_\tau$ so that an allocation A'_τ of maximum social welfare (among those that distribute the particular bundles to the agents) is obtained (in terms of the original valuations). It solves $\text{LP}(A'_\tau, v)$ (for the original valuations) to compute the minimum amount of subsidies that makes A'_τ envy-free. Among all allocations A'_τ, it outputs the one with minimum amount of subsidies. The approximation guarantee of the algorithm is given by the next lemma.

Lemma 8. Given an instance of SMEF that has an allocation that is envy-freeable with an amount of χ as total subsidies, the algorithm computes an allocation that is envy-freeable with total subsidies of at most $\chi + 4mn^2\delta$.

Proof. Let τ be a full tuple such that $\mathcal{A}_\tau$ contains an allocation $O = (O_1, ..., O_n)$ that is envy-freeable with subsidies of χ. Since $\mathcal{A}_\tau$ is non-empty, it is $\mathbf{T}(\tau) = 1$ and let A be the allocation computed by the algorithm as representative of $\mathcal{A}_\tau$. Also, let A' be the allocation that is obtained after redistributing the bundles of A. By Theorem 18, A' is clearly envy-freeable; we will show that the corresponding subsidies are at most $\chi + 4mn^2\delta$. Clearly, the output of the algorithm will be envy-freeable with at most this amount of subsidies.

Let $\sigma \in \mathcal{L}(n)$ be the permutation over $[n]$ such that $A'_j = A_{\sigma(j)}$ for every $j \in [n]$. Let G and H be the envy graphs $\text{EG}(O, v)$ and $\text{EG}(A', v)$, respectively.

We now present the most crucial component of our analysis. It exploits the fact that both O and A belong to class $\mathcal{A}_\tau$ and uses the third statement of Theorem 18.

Lemma 9. For every pair of agents i and j, there exists a (not necessarily simple) path $p(i, j)$ from node $\sigma(i)$ to node $\sigma(j)$ such that

$$\text{wgt}_H(i, j) \leq \sum_{e \in p(i,j)} \text{wgt}_G(e) + 4m\delta$$

In the proof, we will use the following simple claim.

Claim 1. For every agent i and every two bundles B_1 and B_2 such that $\tilde{v}_i(B_1) = \tilde{v}_i(B_2)$, it holds that

$$-|B_2|\delta \leq v_i(B_1) - v_i(B_2) \leq |B_1|\delta. \tag{3.2}$$

Proof. First observe that, by the definition of $\tilde{v}$ and its relation to v, for every agent i and item $g \in M$, it holds that $\tilde{v}_i(g) \leq v_i(g) \leq \tilde{v}_i(g) + \delta$. Hence, for every bundle B, we have

$$\tilde{v}_i(B) \leq v_i(B) \leq \tilde{v}_i(B) + |B|\delta.$$

The claim follows by applying this inequality for bundles B_1 and B_2 and using the fact that $\tilde{v}_i(B_1) = \tilde{v}_i(B_2)$. $\square$

Proof. **Lemma 9** We use the notation σ^{-1} to refer to the inverse permutation of σ, i.e., $\sigma^{-1}(k) = j$ when $k = \sigma(j)$.

Consider the set C that contains edge $(k, \sigma^{-1}(k))$ for every agent k such that $k \neq \sigma^{-1}(k)$. C is either empty (if $k = \sigma^{-1}(k)$ for every agent k) or consists of disjoint directed cycles. For an agent i, if $\sigma^{-1}(i) \neq i$, we denote by C_i the set of nodes that are spanned by the cycle of C that includes node i. Otherwise, we define C_i to contain only node i.

Define the (not necessarily simple) path $p(i,j)$ from node $\sigma(i)$ to node $\sigma(j)$ to contain edge $(k, \sigma(k))$ for every node k in the set C_i besides node i and, if $i \neq \sigma(j)$, the directed edge $(i, \sigma(j))$. For every pair of agents i and j, we have that the weight of the directed edge (i,j) in H is:

$$\begin{aligned}
\mathrm{wgt}_H(i,j) &\leq \mathrm{wgt}_H(i,j) - \sum_{k \in C_i} \mathrm{wgt}_H(k, \sigma^{-1}(k)) \\
&= v_i(A'_j) - v_i(A'_i) - \sum_{k \in C_i} \Big(v_k(A'_{\sigma^{-1}(k)}) - v_k(A'_k) \Big) \\
&= v_i(A_{\sigma(j)}) - v_i(A_{\sigma(i)}) - \sum_{k \in C_i} \big(v_k(A_k) - v_k(A_{\sigma(k)}) \big) \\
&\leq v_i(O_{\sigma(j)}) - v_i(O_{\sigma(i)}) - \sum_{k \in C_i} \big(v_k(O_k) - v_k(O_{\sigma(k)}) \big) \\
&\quad + \left(|A_{\sigma(j)}| + |O_{\sigma(i)}| + \sum_{k \in C_i} |O_k| + \sum_{k \in C_i} |A_{\sigma(k)}| \right) \delta \\
&\leq v_i(O_{\sigma(j)}) - v_i(O_{\sigma(i)}) - \sum_{k \in C_i} \big(v_k(O_k) - v_k(O_{\sigma(k)}) \big) + 4m\delta \\
&= v_i(O_{\sigma(j)}) - v_i(O_i) + \sum_{k \in C_i \setminus \{i\}} \big(v_k(O_{\sigma(k)}) - v_k(O_k) \big) + 4m\delta \\
&= \mathrm{wgt}_G(i, \sigma(j)) + \sum_{k \in C_i \setminus \{i\}} \mathrm{wgt}_G(k, \sigma(k)) + 4m\delta
\end{aligned}$$

$$= \sum_{e \in p(i,j)} \text{wgt}_G(e) + 4m\delta.$$

The first inequality follows since C_i consist of node i only (when $i = \sigma(i)$) or the edges $(k, \sigma^{-1}(k))$ for $k \in C_i$ form a directed cycle of non-positive total weight in H. The second inequality follows by applying Claim 1 (recall that both allocations A and O belong to the class $\mathcal{A}_\tau$ and, hence, $\tilde{v}_\ell(A_q) = \tilde{v}_\ell(O_q)$ for every pair of agents ℓ and q). The third inequality follows since the bundles $A_{\sigma(k)}$ (respectively, O_k) for $k \in C_i$ are disjoint. The equalities are obvious or follow by the definition of the weights. □

Now, let π' and π be the solutions of $\text{LP}(A', v)$ and $\text{LP}(O, v)$, respectively. Hence, $\chi = \text{Sub}(O, v) = \sum_{i=1}^{n} \pi_i$. We will use Lemma 9 to argue that

$$\pi'_i \leq \pi_{\sigma(i)} + 4mn\delta. \tag{3.3}$$

This will yield

$$\text{Sub}(A', v) = \sum_{i=1}^{n} \pi'_i \leq \sum_{i=1}^{n} \left(\pi_{\sigma(i)} + 4mn\delta\right) = \chi + 4mn^2\delta,$$

completing the proof of Lemma 8.

Recall from Theorem 18 that the payment π'_ℓ (respectively, π_ℓ) is equal to the maximum path weight over all simple paths that originate from node ℓ in graph H (respectively, graph G). Let Q_ℓ be the corresponding simple path that is destined for some node s (and originates from node ℓ), i.e., $\pi'_\ell = \sum_{e \in Q_\ell} \text{wgt}_H(e)$. We construct the (not necessarily simple) path P_ℓ from node $\sigma(\ell)$ to node $\sigma(s)$ of G that consists of path $p(i,j)$ for every directed edge (i,j) in the path Q_ℓ. Using Lemma 9, we get

$$\begin{aligned}
\pi'_\ell = \sum_{e \in Q_\ell} \text{wgt}_H(e) &\leq \sum_{e \in Q_\ell} \left(\sum_{e' \in p(e)} \text{wgt}_G(e') + 4m\delta \right) \\
&\leq \sum_{e \in Q_\ell} \sum_{e' \in p(e)} \text{wgt}_G(e') + 4mn\delta \\
&= \sum_{e \in P_\ell} \text{wgt}_G(e) + 4mn\delta.
\end{aligned} \tag{3.4}$$

The second inequality follows since path Q_ℓ is simple (and, hence, contains at most $n-1$ edges). Now, create the simple path P'_ℓ from node $\sigma(\ell)$ to node $\sigma(s)$ by removing the cycles in P_ℓ. Since graph G does not have any directed cycles of positive total weight (by Theorem 18), we have $\mathrm{wgt}_g(P_\ell) \leq \mathrm{wgt}_G(P'_\ell)$. Now, (3.4) yields

$$\pi'_\ell \leq \sum_{e \in P'_\ell} \mathrm{wgt}_G(e) + 4mn\delta,$$

which implies (3.3) since P'_ℓ is a simple path that originates from node $\sigma(i)$ and completes the proof of Lemma 8. $\square$

The running time of the algorithm depends on the number of table entries, the number of steps required for computing each table entry using the recursive relation, the number of steps required to compute a representative allocation for a non-empty allocation class, the redistribution time, and the time required to solve the linear programs.

The dimensions of the table $\mathbf{T}$ are m for the first one that enumerates over all items, and at most $1 + \lfloor \max v/\delta \rfloor = 1 + \frac{4mn^2}{\epsilon}$ for each of the other dimensions. Overall, the size of the table is $\mathcal{O}\left(\left(\frac{m}{\epsilon}\right)^{n^2+1}\right)$. The computation of each table entry using the recursive relation needs the values in n^2 table entries that have previously computed. In a representative allocation, the agent in which each of the m items is allocated requires time n^2 as well, i.e., time $\mathcal{O}(m)$ in total. The redistribution of the bundles can be implemented using a matching computation in a complete edge-weighted bipartite graph that has a node for each agent and for each bundle and the weight of an edge indicates the valuation of an agent for a bundle. As n is constant, this takes constant time. Also, the linear programs have constant size. In general, since n is a constant, it is ignored in the $\mathcal{O}$ notation unless it appears in the exponent.

The above discussion is summarized in the next statement.

Theorem 19. *Let $\epsilon > 0$ be the accuracy parameter used by the algorithm. Given an instance of SMEF consisting of a constant number n of agents with valuations v over a set M of m items that has an envy-freeable allocation using an amount χ of subsidies, the algorithm runs in time $\mathcal{O}\left((m/\epsilon)^{n^2+2}\right)$ and computes an allocation that is envy-freeable using a total subsidy of at most $\chi + \epsilon \max v$.*

3.4 Hardness of approximating SMEF

In this section, we show that approximation guarantees like the one in the statement of Theorem 19 are not possible when the number of agents is part of the input. In particular, we show the following negative result.

Theorem 20. *Approximating SMEF within an additive term of* $3 \cdot 10^{-4} \operatorname{sum} v$ *is NP-hard.*

We prove the theorem by presenting a reduction from Maximum 3-Dimensional Matching (MAX-3DM). An instance of MAX-3DM consists of three disjoint sets of elements $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_n\}$, and $C = \{c_1, c_2, ..., c_n\}$, each of size n, and a set T of m triplets of the form (a_i, b_j, c_k) with $a_i \in A$, $b_j \in B$, and $c_k \in C$. The objective is to compute a disjoint subset of T (or, simply, a 3D matching) of maximum size. The problem is well-known to be NP-hard not only to solve exactly [17] but also to approximate [20].

We will use the inapproximability result of Chlebík and Chlebíková [13], which applies to bounded instances of MAX-3DM in which each element appears in exactly two triplets (i.e., $m = 2n$); we will refer to this restriction of MAX-3DM as MAX-3DM-2. In particular, Chlebík and Chlebíková [13] show that it is NP-hard to distinguish between instances of MAX-3DM-2 with a 3D matching of size at least K and instances of MAX-3DM-2 in which any 3D matching has size at most $K - 0.01n$.[2]

The reduction. We present our reduction and full proof for the case $\chi > 0$. At the end of the section, we discuss the minor modifications required for the case $\chi = 0$. On input an instance of MAX-3DM-2, our reduction constructs in polynomial time an instance of SMEF, in which the minimum amount of susbsidies that can make some allocation envy-free is exactly $\chi(1 + \max\{K - L\})$, where L is the size of the maximum 3D matching in the MAX-3DM-2 instance. Using the result of Chlebík and Chlebíková [13], we will get that it is NP-hard to distinguish between SMEF instances in which the minimum amount of subsidies is at most χ and instances in which it is at least $\chi(1 + 0.01n)$. Hence, SMEF will be proved to be NP-hard to approximate within $0.01n\chi$.

[2]This statement is actually weaker than the one proved by Chlebík and Chlebíková [13]. However, it suffices for our purpose to prove hardness of approximation. Note that we have made no particular attempt to optimize our inapproximabity threshold.

Our construction will be such that $\operatorname{sum} v < 30n\chi$. In this way, we will obtain a hardness of approximating SMEF within an additive term of (at least) $3 \cdot 10^{-4} \operatorname{sum} v$, as desired.

Our reduction is as follows. Given an instance of MAX-3DM-2 consisting of sets of elements A, B, and C, each of size n, and a set of $2n$ triplets T, the instance of SMEF has

- three agents 1, 2, and 3,
- three agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ for every triplet $t \in T$,
- an item A_i for every element $a_i \in A$,
- an item B_i for every element $b_i \in B$,
- an item Γ_i for every element $c_i \in C$,
- three items Δ_t, Z_t, and Θ_t for every triplet $t \in T$, and
- an additional item Λ.

The agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ that correspond to the triplet $t = (a_i, b_j, c_j)$ have valuations 0 for all items besides the items A_i, B_j, Γ_k, Δ_t, Z_t, and Θ_t. Agents 1, 2 have valuation 0 for all items besides item Λ and agent 3 has valuation zero for all items besides item Λ and items Θ_t for $t \in T$. Their remaining valuations are as follows:

	A_i	B_j	Γ_k	Δ_t	Z_t	Θ_t	Λ
1	0	0	0	0	0	0	χ
2	0	0	0	0	0	0	χK
3	0	0	0	0	0	χ	χK
$J_1(t)$	χ	χ	χ	3χ	3χ	0	0
$J_2(t)$	0	0	0	χ	χ	χ	0
$J_3(t)$	0	0	0	0	χ	0	0

Recall that each element belongs to exactly two triplets. Hence, two agents have positive value for item A_i (similarly for items B_j and Γ_k): agents $J_2(t_1)$ and $J_2(t_2)$ such that the triplets t_1 and t_2 contain element a_i (similarly for elements b_j and c_k). It is easy to see that either two or three agents have positive value for each item. For every triplet t, the agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ have total valuation 9χ, 3χ, and χ, respectively. Taking into account that $K \leq n$, we obtain that $\operatorname{sum} v < 30n\chi$.

Lower bound on subsidies. Consider an instance of SMEF constructed by our reduction and let X be an envy-freeable allocation in it. We will first lower-bound the minimum total subsidies that make X envy-free. First observe that X cannot give item Λ to agent 1; in that case, exchanging the bundles of agents 1 and 2 would result to an increase of the social welfare and, hence, X would not be envy-freeable. If X gives item Λ to agent 3, agents 1 and 2 would need subsidies of at least χ and χK, respectively, so that they do not envy agent 3. Hence, $\mathrm{Sub}(A, v) \geq \chi(1 + K)$ in this case.

In the following, we will lower-bound the minimum total subsidies that make X envy-free assuming that item Λ is given to agent 2. Let θ be the number of items Θ_t for $t \in [2n]$ agent 3 gets. Then, agent 3 should be given a subsidy of at least $\chi \max\{K - \theta, 0\}$ so that she does not envy agent 2. Agent 1 needs a subsidy of $\chi \max\{K - \theta, 1\}$ so that she does not envy agents 1 and 2.

For a triplet $t = (a_i, b_j, c_k)$ in the original instance of MAX-3DM-2, we call it *full* if all items A_i, B_j, and Γ_k (which correspond to the elements of the triplet) have been allocated to the agents $J_1(t)$, $J_2(t)$, or $J_3(t)$. Otherwise, we call it *partial.* We call t *supported* if item Θ_t has been allocated to agent $J_2(t)$; otherwise, we call t *unsupported.*

In the next four claims, we lower-bound the total amount of subsidies the agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ of a triplet t need, depending of the type of t.

Claim 2. The agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ of a full and supported triplet t need subsidies of at least $\chi \max\{K - \theta - 2, 0\}$.

Proof. Consider a full and supported triplet t. If agent $J_2(t)$ has value at most 2χ (i.e., getting Θ_t and at most one of the items Δ_t and Z_t), then she needs a subsidy of at least $\chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent 3. If agent $J_2(t)$ has value 3χ by getting both items Δ_t and Z_t in addition to Θ_t, she needs a subsidy of at least $\chi \max\{K - \theta - 3, 0\}$, while then agents $J_1(t)$ and $J_3(t)$ need subsidies of at least $3\chi + \chi \max\{K - \theta - 3, 0\}$ and $\chi + \chi \max\{K - \theta - 3, 0\}$, respectively, so that they do not envy agent $J_2(t)$. In both cases, the total amount of subsidies of the agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ is at least $\chi \max\{K - \theta - 2, 0\}$. □

Claim 3. The agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ of a full and unsupported triplet need subsidies of at least $\chi \max\{K - \theta - 1, 0\}$.

Proof. Consider a full and unsupported triplet t. If agent $J_2(t)$ has value at most χ (i.e., getting at most one of the items Δ_t and Z_t), then she needs a subsidy of at least $\chi \max\{K - \theta - 1, 0\}$ so that she does not envy agent 3. If agent $J_2(t)$ has value 2χ by getting both items Δ_t and Z_t, she needs a subsidy of at least $\chi \max\{K - \theta - 2, 0\}$, while then agents $J_1(t)$ and $J_3(t)$ need subsidies of at least $3\chi + \chi \max\{K - \theta - 2, 0\}$ and $\chi + \chi \max\{K - \theta - 2, 0\}$, respectively, so that they do not envy agent $J_2(t)$. In both cases, the total amount of subsidies of agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ is at least $\chi \max\{K - \theta - 1, 0\}$. □

Claim 4. The agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ of a partial and supported triplet t need subsidies of at least $\chi \max\{K - \theta - 1, 0\}$.

Proof. Let t be a partial and supported triplet. If agent $J_2(t)$ does not get items Δ_t and Z_t, then she gets only a value of χ from item Θ_t and needs a subsidy of at least $\chi \max\{K - \theta - 1, 0\}$ so that she does not envy agent 3.

If agent $J_2(t)$ gets item Δ_t but not item Z_t, she needs a subsidy of $\chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent 3. Then, if agent $J_1(t)$ does not get item Z_t, her value is at most 2χ (from at most two of the items A_i, B_j, and Γ_k) and needs a subsidy of $\chi + \chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent $J_2(t)$. If agent $J_3(t)$ does not get item Z_t, she needs a subsidy of at least $\chi + \chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent $J_2(t)$.

If agent $J_2(t)$ gets item Z_t but not Δ_t, she needs a subsidy of $\chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent 3 and agent $J_3(t)$ needs a subsidy of at least $\chi + \chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent $J_2(t)$.

Finally, if agent $J_2(t)$ gets items Δ_t and Z_t, her value is 3χ and needs a subsidy of at least $\chi \max\{K - \theta - 3, 0\}$ so that she does not envy agent 3. Then, each of agents $J_1(t)$ and $J_3(t)$ need a subsidy of at least $\chi + \chi \max\{K - \theta - 3, 0\}$ so that they do not envy agent $J_2(t)$.

So, the total amount of subsidies the agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ need in each case are at least $\chi \max\{K - \theta - 1, 0\}$. □

Claim 5. The agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ of a partial and unsupported triplet t need subsidies of at least $\chi \max\{K - \theta, 1\}$.

Proof. Let t be a partial and unsupported triplet. If agent $J_2(t)$ gets both items Δ_t and Z_t, she needs a subsidy of $\chi \max\{K - \theta - 2, 0\}$ so that she does not envy agent 3, while agents $J_1(t)$ and

$J_3(t)$ would then need subsidies of at least $4\chi + \chi \max\{K-\theta-2, 0\}$ and $\chi + \chi \max\{K-\theta-2, 0\}$, respectively, so that they do not envy agent $J_2(t)$.

If agent $J_2(t)$ gets only item Δ_t, she needs a subsidy of $\chi \max\{K-\theta-1, 0\}$ so that she does not envy agent 3. Then, the agent who does not get item Z_t among $J_1(t)$ and $J_3(t)$ would need a subsidy of at least $\chi + \chi \max\{K-\theta-1, 0\}$ so that she does not envy agent $J_2(t)$.

If agent $J_2(t)$ gets only item Z_t, she needs a subsidy of $\chi \max\{K-\theta-1, 0\}$ so that she does not envy agent 3, while agent $J_3(t)$ needs a subsidy of at least $\chi + \chi \max\{K-\theta-1, 0\}$ so that she does not envy agent $J_2(t)$.

Finally, if agent $J_2(t)$ gets no item (among Δ_t and Z_t), she needs a subsidy of at least χ so that she does not envy the agents who get items Δ_t and Z_t and a subsidy of at least $\chi \max\{K-\theta, 0\}$ so that she does not envy agent 3.

So, the total amount of subsidies the agents $J_1(t)$, $J_2(t)$, and $J_3(t)$ need in each case are at least $\chi \max\{K-\theta, 1\}$. □

We now denote by L_1, L_2, P_1, and P_2, the number of full and supported, full and unsupported, partial and supported, and partial and unsupported triplets defined by X, respectively. Notice that the full triplets form a 3D matching. Denoting by L the maximum size over all 3D matchings of the MAX-3DM-2 instance, we have $L \geq L_1 + L_2$. Using Claims 2-5, and our observations for agents 1 and 3, we have that the total amount of subsidies X needs to become envy-free is

$$\begin{aligned}
&\mathrm{Sub}(X, v) \\
&\geq \chi \left(L_1 \max\{K-\theta-2, 0\} + L_2 \max\{K-\theta-1, 0\}\right. \\
&\quad + P_1 \max\{K-\theta-1, 0\} + P_2 \max\{K-\theta, 1\} \\
&\quad \left. + \max\{K-\theta, 0\} + \max\{K-\theta, 1\}\right). \qquad (3.5)
\end{aligned}$$

We will distinguish between two cases for $K-\theta$. If $K-\theta \geq 2$, (3.5) yields, $\mathrm{Sub}(X, v) \geq \chi\left(L_2 + P_1 + 2P_2 + 4\right) = \chi\left(2n - L_1 + P_2 + 4\right) \geq \chi(1 + \max\{K-L, 0\})$.

Now, notice that θ, the number of items Θ_t agent 3 gets in X is upper-bounded by the number of unsupported triplets, i.e., $\theta \leq L_2 + P_2$. Thus, if $K-\theta \leq 1$, (3.5) yields $\mathrm{Sub}(X, v) \geq \chi\left(P_2 + K - \theta + 1\right) \geq \chi\left(K - L_2 + 1\right) \geq \chi(1 + \max\{K-L, 0\})$.

We conclude that the minimum amount of subsidies necessary to make X envy-free is at least $\chi(1 + \max\{K-L, 0\})$.

Upper bound on minimum subsidies. We now present our upper bound on the minimum amount of subsidies for envy-freeness. Given a 3D matching $\mathcal{M}$ of maximum size L in the MAX-3DM-2 instance, we will construct an allocation for the SMEF instance and will show that it is envy-freeable with an amount of subsidies equal to $\chi(1+\max\{K-L,0\})$.

For defining the allocation, we partition $T \setminus \mathcal{M}$ in two disjoint sets of triplets T_1 and T_2 of size $2n - \max\{K, L\}$ and $\max\{K-L, 0\}$, respectively.

- For every triplet $t = (a_i, b_j, c_k) \in \mathcal{M}$, agent $J_1(t)$ gets items A_i, B_j, and C_k, agent $J_2(t)$ gets item Δ_t and agent $J_3(t)$ gets item Z_t.

- For every triplet $t = (a_i, b_j, c_k) \notin \mathcal{M}$, let $F(t)$ be the set of items that correspond to the elements of t that have not been included in triplets of $\mathcal{M}$. Note that, due to the maximality of $\mathcal{M}$, $F(t)$ has zero, one, or two elements among A_i, B_j, and Γ_k. For every triplet $t = (a_i, b_j, c_k) \in T_1$, agent $J_1(t)$ gets item Δ_t, agent $J_2(t)$ gets the items in $F(t)$, if any, and item Θ_t, and agent $J_3(t)$ gets item Z_t.

- For every triplet $t = (a_i, b_j, c_k) \in T_2$, agent $J_1(t)$ gets item Δ_t, agent $J_2(t)$ gets the items in $F(t)$, if any, and agent $J_3(t)$ gets item Z_t.

- Agent 3 gets item Θ_t for every triplet $t \in \mathcal{M} \cup T_2$.

- Agent 2 gets item Λ.

- Agent 1 gets no items.

We claim that the allocation above is envy-freeable by assigning a subsidy of χ to agent 1 and a subsidy of χ to agent $J_2(t)$ for every triplet $t \in T_2$ (if any).

Indeed, agent 1 has positive value only for item Λ, which is given to agent 2, who gets no subsidy. Also, no other agent gets a subsidy more than the subsidy χ that is given to agent 1. Hence, agent 1 is not envious. Agent 2 gets item Λ, which is the only item she values positively and much higher than the subsidy given to any other agent. Hence, agent 2 is not envious either. Agent 3 gets exactly $\max\{K, L\}$ items of total value of $\chi \max\{K, L\}$. She does not envy agent 2 who gets item Λ (which agent 3 values for χK) since no subsidy is given to agent 2. Clearly, the value of agent 3 is much higher than the subsidy given to any other agent.

Consider a triplet $t = (a_i, b_j, c_k) \in \mathcal{M}$. Agent $J_1(t)$ has a value of 3χ for the items A_i, B_j, and Γ_k she gets. The remaining items for which she has positive valuation of 3χ have been given to agents $J_2(t)$ and $J_3(t)$, respectively. Since these agents do not get subsidies, agent $J_1(t)$ is not envious of them. Clearly, agent $J_1(t)$ is not envious of any other agent since she has zero value for all other items and no agent gets a subsidy more than χ. Agent $J_3(t)$ gets item Z_t, the only item for which she has positive value and does not envy any other agent since no one gets a subsidy higher than χ. Agent $J_2(t)$ gets a value of χ from item Δ_t and does not envy agent $J_3(t)$ who gets item Z_t or agent 3 who gets item Θ_t as these agents receive no subsidy. Clearly, agent $J_2(t)$ envies no other agent.

Now consider a triplet $t = (a_i, b_j, c_k) \notin \mathcal{M}$. Agent $J_1(t)$ has a value of 3χ for the item Δ_t she gets. The remaining items for which she has positive valuation have been allocated as follows. Item Z_t has been given to agent $J_3(t)$; clearly, agent $J_1(t)$ is not envious of $J_3(t)$ since the latter gets no subsidies. The items in $F(t)$ have been given to agent $J_2(t)$. Again, agent $J_1(t)$ is not envious of $J_2(t)$ since $F(t)$ contains at most two items (which agent $J_1(t)$ values for χ each) and agent $J_2(t)$ gets a subsidy of zero (if $t \in T_1$) or χ (if $i \in T_2$). Clearly, $J_1(t)$ does not envy any other agent. Agent $J_3(t)$ gets item Z_t, the only item for which she has positive value and does not envy any other agent since no one gets a subsidy higher than χ. Agent $J_2(t)$ gets a value of χ either from item Θ_t (if $t \in T_1$) or as subsidy (if $t \in T_2$) and does not envy agent $J_1(t)$ who gets item Δ_t or agent 3 who gets item Θ_t only when $t \in T_2$; recall that these two agents never get subsidies. Again, agent $J_2(t)$ envies no other agent.

Adapting the proof for the case $\chi = 0$. The modification required in our reduction so that it covers the case $\chi = 0$ as well is to remove agent 1 and replace χ with 1 in the definition of valuations. The same reasoning as above gives a minimum amount of subsidies for the SMEF instance of exactly $\max\{K - L, 0\}$, where L is the maximum 3D matching size in the MAX-3DM-2 instance. In this way, we get that SMEF is NP-hard to approximate within $0.01n$ and the construction satisfies $\operatorname{sum} v < 30n$. This yields the desired inapproximability result in the statement of Theorem 18 for the case $\chi = 0$ as well.

3.5 Summary

In this final chapter we presented our contribution to the study of fair division of indivisible goods. We have initiated the study of the optimization problem SMEF. The apparent open problem that deserves investigation is to close the gap between the trivial approximation guarantee of $n-1$ in Section 3.2 and our negative result for super-constant numbers of agents. Interestingly, an advantage the trivial algorithm has is that the particular payments incentivize the agents to report their valuations truthfully. What is the possible approximation guarantee that can be obtained for SMEF by truthful algorithms? Unfortunately, a simple application of Myerson's [26] characterization in single-item settings indicates that no approximation guarantee better than $n-1$ is possible.

Chapter 4

Epilogue

Fair division of indivisible goods is a research area that has gathered the attention of many scientists through the last years. As a subfield of computational social choice, an area that arose at the edge of algorithmic game theory and social sciences, it combines subjects, concepts and ideas from algorithms, economics, computational complexity and social sciences. Like every newborn field in science it has attracted the attention of many scientists and students.

In this thesis we studied problems of fair division of indivisible goods. The choice of the presented concepts and results was based on papers we studied and emphasized on, that helped us understand the basic problems and contributed to our results as they were presented in the final chapter. We emphasized on approximations of envy-freeness like the $EF1$ and EFX notions and presented some important theoretic results on them.

Throughout the course of this master thesis, many new results have been proved and are being discovered every year. We tried to keep pace of the scientific progress. During a 2 year period there has been significant scientific progress regarding the problems and subjects of our study. This proves that the field has attracted a lot of interest. We stated the that EFX notion introduced in [9] has no proof about its guaranteed existence. Also in [28] an approximation algorithm for it was presented. A very recent result by Chaudhury et al [10], states the existence of the EFX notion under additive valuations for three agents. Also very recent work on fairness approximations has been made in [11].

In chapter 2 we presented a parallel study framework of fair division of indivisible goods. In distributed fair division the agents are behaving independently like autonomous agents and conduct deals to increase their utility. The main characteristic of this framework is the lack of central authority to allocate the goods to the agents. We focused on negotiations that use payments to enable envy-freeness and finally let the system converge to efficient and envy-free allocations. This study area has many extensions and implementations in artificial intelligence.

Finally we presented our results regarding fair division using subsidy. We examined the problem in the subsidy model, a variation of the monetary side payments model. We examined the problem of approximating and computing the minimum subsidy required for an allocation so as to be envy-free. A more detailed presentation of these results can be found in [8].

Many open questions have yet to be answered. Among them is whether EFX exists for every instance under additive valuations. Equivalence of subsidy and monetary side payments model is another open question of some interest. Theory for envy-freeness on graphs and efficient allocations when the agents form a network topology is also a research topic with many implementations in artificial intelligence. Finally better approximation algorithms for the proposed fairness notions is always a research target of computer scientists.

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