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Generalized interval-based polynomial approximations to functions in applied mechanics by using the method of quantifier elimination

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Abstract The method of quantifier elimination constitutes an interesting computational approach in computer algebra already implemented in few computer algebra systems. In applied mechanics, this method was already used for the determination of ranges of functions. Here the application of the same method, quantifier elimination, is generalized to the determination of generalized interval-based polynomial approximations to functions again in applied mechanics. The main idea behind the present application is the use of linear interval enclosures for the approximation to functions and, more generally, the use of parameterized solutions to parametric interval systems of linear algebraic equations. This idea is mainly due to Lubomir V. Kolev. Here the present method is at first applied to two simple examples concerning (i) a rational function and (ii) the exponential function with their variables lying in intervals. Next, the same method is also applied to functions in applied-mechanics problems with variables also lying in intervals: (i) the problem of a beam on a Winkler elastic foundation with related function the dimensionless deflection of the beam, (ii) the problem of free vibrations of an oscillator with critical damping with related function the dimensionless displacement of the oscillator and (iii) the problem of a seven-member truss with related functions the nodal displacements. In this application, the stiffness of a bar is an uncertain, interval variable and, moreover, the classical perturbation method is also used. From the present results it is concluded that the method of quantifier elimination constitutes a useful tool for the derivation of simple parameterized interval-based polynomial approximations to functions in applied mechanics.

Keywords Intervals · Interval parameters · Uncertainty · Ranges · Polynomial approximations · Interval-based approximations · Generalized approximations · Parameterized approximations · Interval enclosures · Beams · Winkler foundation · Oscillators · Critical damping · Displacements · Trusses · Perturbations · Quantified formulae · Quantified/free variables · Quantifier elimination · Quantifier-free formulae · Quantifiers · Symbolic computations · Computer algebra systems

1. Introduction

1.1. Computer algebra systems, symbolic computations and quantifier elimination

Computer algebra systems are computer programs that perform both symbolic and numerical computations. Such systems constitute an interesting and powerful tool in applied mechanics as well as in a large number of research fields since the sixties. The best known of these systems are *ALTRAN* (1965), *REDUCE* (1966), *Macsyma* (1968), *muMath* (1978), *Maple* (1982), *Derive* (1988) and *Mathematica* (1988). A review on symbolic computations concerning applied and structural mechanics (published in 2003) was prepared by Pavlović [1]. In his research, the author used *Derive*, *REDUCE*, *Maple* and *Mathematica*. The present results are derived by using *Mathematica* [2].

On the other hand, quantifier elimination in elementary real algebra constitutes an interesting and rather recent computational tool in computer algebra. Its aim is simply the elimination of the two well-known quantifiers, i.e. (i) the universal quantifier $\forall$ (for all) and (ii) the existential quantifier $\exists$ (exists) in formulae including quantified variables (quantified formulae) with at least one of these two quantifiers (e.g. $\forall x$ or $\exists x$). In this way, completely equivalent (without approximations) formulae from the mathematical and logical points of view are derived. These formulae are called QFFs (quantifier-free formulae) because they are free from these two quantifiers ($\forall$ and $\exists$) and they include only the free variables that are present in the quantified formula on which quantifier elimination has been performed. An extensive bibliography on the applications of quantifier elimination in elementary real algebra was prepared by Ratschan [3] in 2012. With respect to the algorithms used in quantifier elimination two of these algorithms are very efficient and extensively used:

- The most popular and simultaneously general-purpose algorithm for quantifier elimination is CAD (cylindrical algebraic decomposition). This is a very well-known and useful algorithm and it was devised by Collins in 1973. (At first, it was presented at a symposium held at Carnegie-Mellon University.) Its first official and complete publication by Collins appeared in 1975 [4]. The standard book on quantifier elimination and CAD is still the book edited by Caviness and Johnson [5] published (with some delay) in 1998. This book was based on a related symposium held at the Research Institute for Symbolic Computation in Linz, Austria (RISC-Linz) in October 1993 for the celebration of the 20th anniversary of CAD, but it also includes all the related fundamental research results going back to the fundamental original results on quantifier elimination by Tarski (during the period 1930–1951) at first officially published in 1948 and, next, in 1951. Additionally, a very large number of interesting research results on CAD by many authors is available in the literature; see, e.g., Refs. [6, 7]. The best recent implementation of CAD seems to be that by Strzeboński in *Mathematica* [2]. (This is the implementation that will also be used here.) But on the other hand, unfortunately, from the negative point of view it should be mentioned that quantifier elimination for real variables has a doubly-exponential computational complexity [8]. This result was proved by Davenport and Heintz [8] and is now very well known and, evidently, applicable to CAD as well. Naturally, this result constitutes a serious disadvantage of the method of quantifier elimination and, hence, a significant obstacle to its wide application especially to quantified formulae with a large total number of variables, i.e. both free and quantified variables.
- A second and completely different popular method for quantifier elimination is the method of virtual substitution based on the results mainly by Weispfenning [9–11]. Exactly as CAD (cylindrical algebraic decomposition), virtual substitution has also been already successfully applied to many quantifier elimination problems, but it is generally preferable to CAD only in cases of linear and quadratic polynomials appearing in the quantified formulae.

Both CAD and virtual substitution (as well as some additional algorithms for quantifier elimination) are available since 2003 in the implementation of quantifier elimination in *Mathematica* [2] made mainly by Strzeboński. The selection of the preferable algorithm between CAD and virtual substitution is generally made automatically by *Mathematica* [2] itself, but, alternatively, it can easily be made through the use of an appropriate command. The use of quantifier elimination commands in *Mathematica* [2] is mainly described in the Wolfram monograph [12] as well as in the related pages in the book by Trott [13, pp. 60–78], which is devoted to symbolic computations.

At this point we can mention that from the computational point of view quantifier elimination is a rather difficult task and, therefore, the implementations of the related algorithms (mainly CAD and virtual substitution) in computer algebra systems are rather recent and very few in number. The implementations of quantifier elimination beyond that included in *Mathematica* [2] known to the author are the following three quantifier elimination packages:

- QEPCAD (now QEPCAD B): this classical and famous package of the *SACLIB* library (the first package that performs quantifier elimination) is based on partial CAD (cylindrical algebraic decomposition) and it was prepared mainly by Hong under the guidance of Collins, but with several additional contributors,
- REDLOG: this package of *REDUCE* is mainly based on the method of virtual substitution (and to a less extent on partial CAD) and it was prepared by Dolzmann and Sturm,
- SyNRAC: this package of *Maple* is based on the methods of CAD, virtual substitution and Sturm–Habicht sequences and it was prepared mainly by Anai and Yanami.

Following many researchers in various research fields, since 1994 the author has been interested in the application of the method of quantifier elimination to several problems of applied mechanics (see, e.g., Refs. [14, 15]). Much more recent applied-mechanics results by the author based on the same computational method, quantifier elimination, can be found in Refs. [16–27]. On the other hand, the direct application of CAD (cylindrical algebraic decomposition) to an interesting applied-mechanics problem (the computation of optimal solutions to truss problems in structural mechanics) was recently successfully made by Charalampakis and Chatzigiannelis [28].

1.2. Interval analysis and its applied-mechanics applications

As far as interval analysis is concerned, its modern era is generally considered to have begun in 1959 with the publication of the related famous results by Moore and his collaborators; see, e.g., Refs. [29–31]. Previous related important results are due to Sunaga [32] in 1958 and to few other authors as well. An interesting recent extensive bibliography on interval computations and reliable computing including 784 entries was prepared by Beebe, Kearfott and Kreinovich [33] in 2017.

Beyond mathematics interval analysis has also proved to be a very useful computational tool in applied and computational mechanics during the last thirty years. Among an extremely large number of related interesting publications here we can make reference e.g. to the papers (in chronological order) by Dimarogonas [34], Qiu, Chen and Song [35], Qiu, Chen and Elishakoff [36], Qiu and Elishakoff [37], Kulpa, Pownuk and Skalna [38], McWilliam [39], Dessombz, Thouverez, Laîné, and Jézéquel [40], Skalna [41, 42], Elishakoff and Ohsaki [43] (book), Elishakoff and Miglis [44, 45], Wang and Qiu [46], Gabriele and Varano [47], Behera [48], Santoro, Muscolino and Elishakoff [49], Sofi, Muscolino and Elishakoff [50], Qiu and Wang [51], Zieniuk, Kapturczak and Kużelewski [52], Elishakoff, Gabriele and Wang [53], Popova [54–56], Chakraverty, Hladík and Behera [57], Muscolino, Sofi and Giunta [58], Su, Zhu, Wang, Li and Yang [59], Popova and Elishakoff [60], Popova [61, 62], Sofi, Romeo, Barrera and Cocks [63], Muscolino and Santoro [64], Sofi, Muscolino and Giunta [65], Faes and Moens [66–68], Behera and Chakraverty [69], Popova and Elishakoff [70], Muhanna and Shahi [71], Santoro, Failla and Muscolino [72], Rao and Alazwari [73], Ni and Jiang [74], Dinh-Cong, Van Hoa and Nguyen-Thoi [75] and Popova [76].

We can also mention that, evidently, applied-mechanics applications (mainly in truss problems in structural mechanics) frequently appear as applications in more mathematical interval-analysis publications concerning parametric interval systems of linear algebraic equations; see, e.g., the recent book by Skalna [77] on parametric interval algebraic systems and the recent papers by Kolev [78], Skalna and Hladík [79] and Popova [80] including applications to truss problems with the book by Skalna [77] also including an application to a steel frame again in structural mechanics.

Finally, as far as *Mathematica* [2] is concerned, the commands devoted to interval arithmetic are described by Keiper [81]. Moreover, a *Mathematica* package, the package `directed.m`, concerning directed interval arithmetic and extending the interval capabilities of *Mathematica* was prepared by Popova and Ullrich [82] in 1996. Another interesting *Mathematica* package, the package `IntervalComputations‘LinearSystems‘`, devoted to the solution of parametric/nonparametric systems of linear equations with uncertainties was also prepared by Popova [83] in 2000–2004.

1.3. Relationship between interval analysis and quantifier elimination

It is very well known that quantifiers and quantifier elimination are strongly related to interval analysis. This situation is obvious and natural by taking into account the fact that several problems in interval analysis (including the solution sets of parametric or non-parametric interval systems of linear algebraic equations) are expressed in terms of formulae with universally and/or existentially quantified variables. There is a very large number of related results in the interval literature. Among these results here we make reference to modal intervals by Sainz *et al.*, see, e.g., the book [84], as well as to the results by Grandón and Neveu [85], Grandón and Goldsztejn [86], Goldsztejn [87] and Khanh and Ogawa [88]. On the other hand, Elishakoff, Gabriele and Wang [53] repeatedly used quantifiers in their study of the generalized Galilei problem [53], e.g. at the end of Section 2 there [53, p. 1207]. Thus, they succeeded in providing a physical meaning to a simple interval equation.

At this point we can also note that the implementation of quantifier elimination in *Mathematica* was successfully used by Popova [89] as well as by Popova and Krämer [90] for the characterization of solution sets of parametric interval systems of linear algebraic equations. But, unfortunately, the derived results required too much CPU (central processing unit) time [89] in the computer used or they contained a very large number of logical expressions [90] in comparison with the efficient methods by the same authors for the same computational tasks.

In ten recent technical reports [18–27], the author also combined quantifier elimination (by using its implementation in *Mathematica* in the first nine of these reports [18–26] and REDLOG in *REDUCE* in the tenth report [27]) with interval analysis in the following problems, almost all of which concern applied mechanics (with the exception of the report [21]): (i) the computation of ranges of functions appearing in problems of applied mechanics [18], (ii) the determination of ranges of values of stress concentration factors in plane elasticity, more explicitly in notch and hole problems [19], (iii) similarly, the determination of ranges of values of stress intensity factors at crack tips in plane elasticity problems related to fracture mechanics [20], (iv) the derivation of sharp enclosures of the real roots of the classical parametric quadratic equation but with only one interval coefficient [21], (v) the determination of sharp bounds for intervals in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters [22], (vi) the derivation of symbolic intervals in simple problems of applied mechanics [23], (vii) the computation of intervals in three direct and inverse applied mechanics problems, more explicitly, a classical beam problem, a problem of a beam on a Winkler elastic foundation and the problem of free vibrations of the classical damped harmonic oscillator with critical damping [24], (viii) the determination of intervals (ranges) for the resultants of interval forces satisfying existentially and/or universally quantified formulae [25], (ix) the determination of intervals (ranges) for the unknowns in systems of parametric interval linear equilibrium equations in applied mechanics including the case of appearance of both the universal and the existential quantifiers in the quantified formulae concerning the studied applied mechanics problems [26] and, recently, (x) the computation of intervals in classical beam problems using the computational methods of finite differences and of finite elements [27].

1.4. Aim, origin and contents of this technical report

The present results concern generalized interval-based polynomial approximations (here interval enclosures) $\tilde{f}(x,r)$ to functions $f(x)$ (with their variables x lying in intervals X, i.e. $x \in X$) appearing in problems of applied mechanics. These approximations $\tilde{f}(x,r)$ are based on ordinary polynomial approximations $\tilde{f}(x)$ to the functions $f(x)$ under consideration, but they are supplemented by an interval parameter $r \in R$, where R is the related interval (the range of r). This interval can be computed by using the method of quantifier elimination on the basis of the universally–existentially quantified formula

$$\forall x \in X \; \exists r \in R \; \text{ such that } \; f(x) = \tilde{f}(x,r), \tag{1}$$

which (after quantifier elimination) gives us the bounds R_1 (lower bounds) and R_2 (upper bounds) of the sought interval R and, therefore, essentially, this interval $R = [r_1, r_2]$ (with $R_1 \le r_1 = \min r$ and $R_2 \ge r_2 = \max r$). In practice, the above quantifier elimination task can also be achieved (clearly, in a computational much easier way) by using the simpler and only existentially quantified formula

$$\exists x \in X \; \text{ such that } \; f(x) = \tilde{f}(x,r), \tag{2}$$

which (again after quantifier elimination) directly gives us the sought interval $R = [r_1, r_2]$.

Naturally, several such approximations $\tilde{f}(x,r)$ can be computed for each function $f(x)$, not only approximations of the simple form $\tilde{f}(x) + r$, where the interval parameter r ($r \in R$) appears simply as an additive constant. Therefore, we call all of these approximations (for convenience including the special case of the simple form $\tilde{f}(x) + r$) generalized interval-based polynomial approximations.

The origin of the present results is the existing literature on interval-based approximations to functions particularly linear interval enclosures. Here an emphasis is put on the paper by Kolev concerning the automatic computation of linear interval enclosures [91]. Previous results referenced in this paper by Kolev [91] include those by Krawczyk and Neumaier on interval slopes for rational functions [92], Zuhe and Wolfe on interval enclosures using slope arithmetic [93] and Kolev on the use of interval slopes for the irrational part of factorable functions [94]; see also the related papers by Kolev [95] and Neumaier [96]. Additionally, parameterized solutions (frequently called p-solutions) have been used in parametric interval systems of linear algebraic equations mainly by Kolev, see, e.g., [78, 97–100], as well as by Skalna and Hladík [79, 101] and by Popova [80, 102], who recently proposed a generalized type of such parameterized solutions [76]. On the other hand, Collins used quantifier elimination in Solotareff's approximation problem [103]; see also the paper by Vajda [104, Section 3, pp. 4–8] including the same problem with the use of *Mathematica* [2].

From a different perspective the present results generalize the application of the computational method of quantifier elimination with the help of *Mathematica* [2] from the computation of ranges of functions in applied mechanics (see Ref. [18]), which, naturally, are very rough interval approximations to these functions, to the computation of interval-based polynomial approximations to the same functions, which, evidently, are much more accurate approximations to the same functions.

Finally, as far as the contents of the present technical report are concerned, beyond the present introductory section, Section 1, the present technical report is organized as follows:

- In Section 2, we use the present method based on quantifier elimination for the derivation of a linear (but based on an interval) approximation to a rational function. This application was already successfully studied by Kolev [91, Section 2, Example 2.1, pp. 19–20] by using a different method.
- In Section 3, we derive interval-based polynomial approximations to the exponential function.
- In Section 4, we similarly study the problem of a beam lying on a Winkler elastic foundation.
- In Section 5, we study the problem of free vibrations of an oscillator with critical damping.
- In Section 6, by using the same approach and, additionally, the popular method of perturbations we study a problem concerning a classical seven-member truss with an interval stiffness parameter.
- In Section 7, we state the conclusions drawn from the present results concerning generalized interval-based polynomial approximations to functions and we also include a brief related discussion.

2. A simple interval-based polynomial approximation to a rational function

As a first application of the present approach, which is based on quantifier elimination and the use of *Mathematica* [2], we consider the approximation to the simple rational function

$$f(x) = x - \frac{10}{x + \dfrac{2}{x}} = \frac{x(x^2-8)}{x^2+2} \quad \text{with} \quad x \in X = [1,3] \tag{3}$$

already studied by Kolev [91, Section 2, Example 2.1, pp. 19–20]. (Kolev also mentions that this example was taken from the paper by Krawczyk and Neumaier [92, Section 4, Examples 1 and 2, pp. 611–613].) The method used by Kolev is described in the aforementioned reference [91, p. 19] and it is based on the interval-based approximation [91, p. 19, Eqs. (2.1) and (2.2)]

$$F(x) = ax + B \tag{4}$$

to the rational function $f(x)$ in Eqs. (3). ($F(x)$ is a linear interval enclosure of $f(x)$, i.e. $f(x) \in F(x)$.) In the above expression of the approximation $F(x)$ to $f(x)$, the constant a is deterministic and it denotes the interval slope [91, p. 19, Eq. (2.4)] of the rational function $f(x)$ in the interval $X = [1,3]$ (with $x \in X$), i.e.

$$a = \frac{f(3)-f(1)}{3-1} = \frac{f(3)-f(1)}{2} = \frac{43}{33} \approx 1.303030303 \tag{5}$$

and B is an appropriate interval $[B_1, B_2]$ to be determined. The determination of the interval B is based on the zeros x_k of the function $d(x) - a$ in the present interval $[1,3]$ of the variable x, where the function $d(x)$ denotes the first derivative of $f(x)$, and it is clearly described by Kolev [91, p. 19].

Here we also adopt the interval-based approximation $F(x)$ in Eq. (4) to the rational function $f(x)$ in Eq. (3) ($F(x)$ is a linear interval enclosure of $f(x)$ as has been already mentioned), but we use a computer algebra system, *Mathematica* [2], instead and, more explicitly, the powerful implementation of quantifier elimination in *Mathematica* by Strzeboński. The present approach in this introductory application is based on the universally–existentially quantified formula (1) taking here the form

$$\forall x \in X = [1,3] \ \exists b \in B = [B_1, B_2] \ \text{such that} \ f(x) = ax + b, \tag{6}$$

where our initial rational function $f(x)$ is given by Eqs. (3) and the interval slope a in the interval-based approximation $F(x) = ax + B$ in Eq. (4) to the rational function $f(x)$ in Eqs. (3) is given by Eqs. (5). The related quantifier elimination command in *Mathematica* [2] has the form

```
Reduce[ForAll[x,1≤x≤3,Exists[b,B1b≤b≤B2b,f[x]==a x+b]],
       {B1b,B2b},Reals]                                    [c1]
```

and the resulting QFF (quantifier-free formula) has the form

$$B_{1b} \leq r_{4,1} \wedge B_{2b} \geq -\frac{40}{11} \quad \text{and numerically} \quad B_{1b} \leq -3.994121926 \wedge B_{2b} \geq -3.636363636, \tag{7}$$

where $r_{4,1}$ denotes the first real root of the quartic (here also biquadratic) polynomial

$$p_4(s) := 2371842s^4 - 64713825s^2 + 428750000. \tag{8}$$

Evidently, the related interval

$$B = [B_1, B_2] = [-3.994121926, -3.636363636], \tag{9}$$

i.e. the interval B in Eq. (4), coincides with the interval B computed by Kolev [91, p. 19, Eq. (2.8)] in the same application by using his method. Kolev's method is different from the present one since it is not based on the use of quantifier elimination implemented in a computer algebra system here in *Mathematica* [2] (implementation of several quantifier elimination algorithms by Strzeboński).

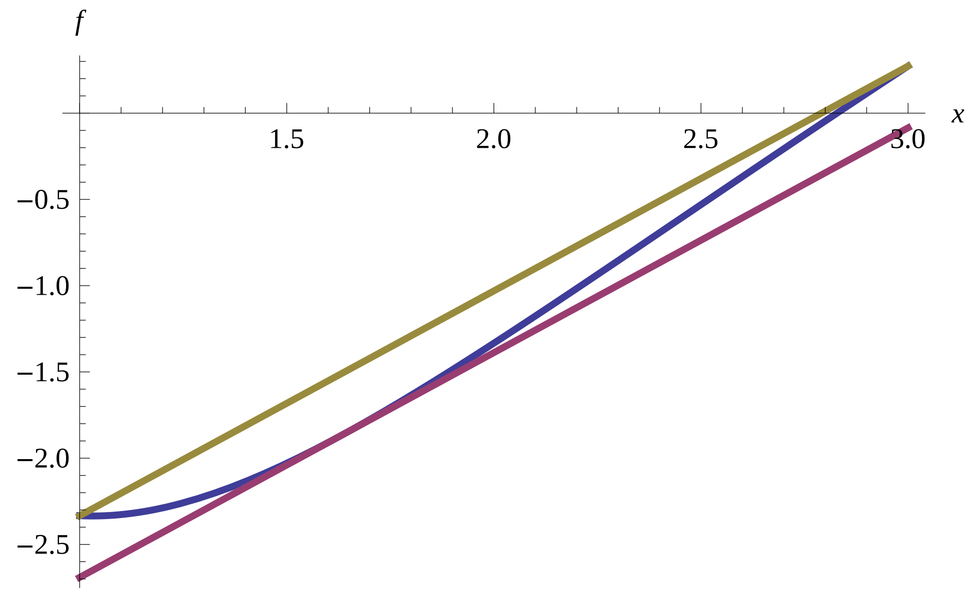

Fig. 1. The rational function $f(x)$ in Eqs. (3) and its two bounds determined by the related interval-based function $F(x) = ax + B$ in Eq. (4) with the interval slope a given by Eqs. (5) and the interval $B = [B_1, B_2]$ given by Eq. (9) as it has been computed here by the method of quantifier elimination.

The main disadvantage of the present approach based on the universally–existentially quantified formula (6) is that it computes the lower bounds B_{1b} and the upper bounds B_{2b} of the endpoints B_1 and B_2, respectively ($B_{1b} \le B_1$ and $B_{2b} \ge B_2$) of the interval $B = [B_1, B_2]$ in Eq. (4). Additionally, the quantified formula (6) includes both a universally quantified variable, the variable x ($\forall x$), and an existentially quantified variable, the variable b ($\exists b$), i.e. two quantified variables.

A computationally simpler, but essentially equivalent as far as the resulting interval $B = [B_1, B_2]$ is concerned, quantified formula, which is now only existentially quantified, has the simple form (2) written here as

$$\exists x \in X = [1, 3] \ \text{ such that } \ f(x) = ax + b, \tag{10}$$

where the variable x is now existentially quantified, the variable b is not quantified any more and the two free variables B_1 and B_2 in the quantified formula (6) do not appear any more. The related quantifier elimination command is now much simpler than the command [c1] having the form

```
Reduce[Exists[x, 1 ≤ x ≤ 3, f[x] == a x+b], Reals]          [c2]
```

The resulting QFF (quantifier-free formula) now takes the simple form

$$r_{4,1} \le b \le -\frac{40}{11} \quad \text{and numerically} \quad -3.994121926 \le b \le -3.636363636, \tag{11}$$

which is more convenient than the QFF (7) and it directly gives the interval $B = [B_1, B_2]$ in Eq. (4) instead of the lower bounds B_{1b} and the upper bounds B_{2b} of its endpoints B_1 and B_2, respectively.

Finally, it should be mentioned that as was expected, the computation of the second QFF, the QFF (11), required much less CPU (central processing unit) computation time, about 0.016 s (seconds), than the computation of the first QFF, the QFF (7), which required more time, about 0.203 s. (These times were computed by using the `Timing` auxiliary command of *Mathematica* [2] and we note that as usual they vary slightly for repetitions of the same quantifier elimination computations.)

For a better (and now visual) understanding of the present interval-based approximation $F(x)$ in Eq. (4) (coinciding with that computed by Kolev [91, p. 19, Eqs. (2.1) and (2.8)]) to the rational function $f(x)$ in Eqs. (3) in Fig. 1 we display $f(x)$ as well as its two bounds (linear enclosures, straight lines) $F_1(x) := ax + B_1$ (for $B = B_1 = -3.994121926$, lower bound, lower straight line in Fig. 1) and $F_2(x) := ax + B_2$ (for $B = B_2 = -3.636363636$, upper bound, upper straight line in Fig. 1).

3. Generalized interval-based polynomial approximations to the exponential function

3.1. Introduction to interval-based polynomial approximations to the exponential function

As a second application of the present approach based on quantifier elimination here we consider the classical exponential function $e^x \equiv \exp x$. Unfortunately, generally, the present computational method of quantifier elimination for real variables is applicable only to polynomial and rational functions and, hence, it is generally inapplicable to transcendental functions such as the exponential function e^x. For this reason instead of the exponential function e^x here we will use its approximation by a polynomial of the 25th degree. Here we are interested in the interval $X = [1,2]$ (with $x \in [1,2]$). Hence, we found it convenient to use the Taylor-series approximation to e^x that is based on the midpoint $x_m = (1+2)/2 = 1.5$ of X with terms up to $n = 25$. All the present computations are performed with the help of *Mathematica* [2]. The related simple series command is

```
exp[x_] = Collect[Series[Exp[x], {x,1.5,25}]//Normal, x]//N                    [c3]
```

and the obtained approximation is denoted by the symbol `exp[x]` in *Mathematica* contrary to the symbol `Exp[x]` used in *Mathematica* for the exponential function $e^x \equiv \exp x$ itself. The resulting 25th-degree polynomial that will be used here instead of the exponential function e^x has the form

$$\begin{aligned}\exp_{25} x = {} & 2.88932\cdot 10^{-25}x^{25} - 3.61165\cdot 10^{-24}x^{24} + 1.08350\cdot 10^{-22}x^{23} + 2.49204\cdot 10^{-22}x^{22} \\ & + 2.39859\cdot 10^{-20}x^{21} + 3.87132\cdot 10^{-19}x^{20} + 8.32550\cdot 10^{-18}x^{19} + 1.55812\cdot 10^{-16}x^{18} \\ & + 2.81262\cdot 10^{-15}x^{17} + 4.77918\cdot 10^{-14}x^{16} + 7.64723\cdot 10^{-13}x^{15} + 1.14707\cdot 10^{-11}x^{14} \\ & + 1.60590\cdot 10^{-10}x^{13} + 2.08768\cdot 10^{-9}x^{12} + 2.50521\cdot 10^{-8}x^{11} + 2.75573\cdot 10^{-7}x^{10} \\ & + 2.75573\cdot 10^{-6}x^{9} + 0.0000248016x^8 + 0.000198413x^7 + 0.00138889x^6 \\ & + 0.00833333x^5 + 0.0416667x^4 + 0.166667x^3 + 0.5x^2 + x + 1. \end{aligned} \tag{12}$$

The resulting accuracy is very satisfactory. In fact, for the difference $d_e(x) := \exp x - \exp_{25} x$ we directly find that it is negligible, more explicitly

$$d_e(1) = 4.44089\cdot 10^{-16}, \quad d_e(2) = 2.66454\cdot 10^{-15}, \quad d_e(1.5) = 1.77636\cdot 10^{-15} \tag{13}$$

at the two endpoints $x_1 = 1$ and $x_2 = 2$ and at their midpoint $x_m = 1.5$ of our present interval $[1,2]$, in which we are interested here. Therefore, here we will simply use $\exp_{25} x \approx \exp x$ instead of $\exp x$.

Now we proceed to our present computational task of deriving generalized interval-based approximations to the exponential function $\exp x \approx \exp_{25} x$. We will base our results on the Taylor-series approximation $\exp_3 x$ to $\exp x$ again based on the midpoint $x_m = 1.5$ of the selected interval $X = [1,2]$ (with $x \in [1,2]$). This approximation $\exp_3 x \approx \exp x$ is obtained by using the command

```
exp3[x_] = Collect[Series[Exp[x], {x,1.5,3}]//Normal, x]//N                    [c4]
```

(completely analogous to the previous command [c3]) and it has the form of the cubic polynomial

$$\exp_3 x = 0.746948x^3 - 1.12042x^2 + 2.80106x + 0.280106. \tag{14}$$

Here our intention is simply to improve the above cubic polynomial approximation $\exp_3 x$ so that we can be sure that the real value $\exp x$ of the exponential function lies in the interval defined by the interval-based improvement of $\exp_3 x$ (to be derived below) for any particular value of $x \in [1,2]$. To this end, we selected to use one of the following five possible interval-based cubic polynomials:

$$e_0(x) = \exp_3 x + r = 0.746948x^3 - 1.12042x^2 + 2.80106x + 0.280106 + r, \tag{15}$$

$$e_1(x) = \exp_3 x + rx = 0.746948x^3 - 1.12042x^2 + (r + 2.80106)x + 0.280106, \tag{16}$$

$$e_2(x) = \exp_3 x + rx^2 = 0.746948x^3 + (r - 1.12042)x^2 + 2.80106x + 0.280106, \tag{17}$$

$$e_3(x) = \exp_3 x + rx^3 = (r + 0.746948)x^3 - 1.12042x^2 + 2.80106x + 0.280106, \tag{18}$$

$$e_r(x) = (1+r)\exp_3 x = (1+r)(0.746948x^3 - 1.12042x^2 + 2.80106x + 0.280106) \tag{19}$$

because of Eq. (14) for the cubic polynomial approximation $\exp_3 x$ to $\exp x$ that we adopted here. In the above five cubic polynomials $e_i(x)$ (with $i = 0, 1, 2, 3, r$), the symbol r denotes a parameter lying in an appropriate interval R (naturally, generally different for each of these polynomials) to be computed below ($r \in R$). This will be achieved by using the method of quantifier elimination that we continuously use here with its implementation in the computer algebra system *Mathematica* [2].

3.2. Computation of the intervals in the generalized interval-based polynomial approximations

We begin with the first interval-based cubic polynomial, i.e. the polynomial $e_0(x)$ in Eq. (15). For this polynomial the following quantified formula should be satisfied on the interval $X = [1, 2]$:

$$\forall x \in X = [1,2] \ \exists r \in R = [r_1, r_2] \ \text{such that} \ \exp_{25} x = e_0(x). \tag{20}$$

This formula includes both the universal quantifier $\forall$ (for all) for the quantified variable x and the existential quantifier $\exists$ (exists) for the quantified variable r as well as the two free variables r_1 and r_2, which are the endpoints of the interval (the range) $R = [r_1, r_2]$ (to be determined) of the quantified variable r. Here we will perform quantifier elimination and we will eliminate both these quantifiers $\forall$ and $\exists$ as well as the related quantified variables x and r, respectively. Hence, by using the following quantifier elimination command corresponding to the above quantified formula (20):

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, r1b ≤ r ≤ r2b, exp[x] == e0[x]]],
        {r1b,r2b}, Reals]//Chop//Timing                                    [c5]
```

we directly find the QFF (quantifier-free formula)

$$r_{1b} \leq 0 \wedge r_{2b} \geq 0.0129428 \tag{21}$$

for the two bounds r_{1b} (lower bounds) and r_{2b} (upper bounds) of the interval (the range) R of r. Hence, on the basis of the bounds r_{1b} and r_{2b}, the resulting interval R appearing in Eq. (15) is

$$R = [r_1, r_2] = [0, 0.0129428]. \tag{22}$$

Therefore, the cubic polynomial $e_0(x)$ in Eq. (15) can be written in its final interval-based form

$$e_0^{\mathrm{I}}(x) = 0.746948x^3 - 1.12042x^2 + 2.80106x + 0.280106 + [0, 0.0129428]. \tag{23}$$

Then we have the following universally quantified formula (an approximate formula with the approximation simply due to the facts that we used $\exp_{25} x$ in Eq. (12) instead of $e^x \equiv \exp x$ itself and, moreover, evidently, that there exist round-off errors in the present numerical computations):

$$\forall x \in X = [1,2] \ \text{it holds true that} \ e^x \equiv \exp x \in e_0^{\mathrm{I}}(x). \tag{24}$$

Unfortunately, the required computational time (CPU time, central processing unit time) for the above quantifier elimination has been very large, about 60 s (seconds), in *Mathematica* [2]. In order to significantly reduce this excessive computational time, we can work separately for the two bounds r_{1b} and r_{2b} of the endpoints of the sought interval $R = [r_1, r_2]$ of r, i.e. at first with the lower bounds r_{1b} and, next, with the upper bounds r_{2b} of this interval $R = [r_1, r_2]$ (with $r \in R = [r_1, r_2]$). In this way, using the quantifier elimination command for the lower bounds r_{1b}

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, r ≥ r1b, exp[x] == e0[x]]], Reals]
        //Chop//Timing                                                      [c6]
```

in just about 0.5 s we find the simple QFF (quantifier-free formula)

$$r_{1b} \leq 0. \tag{25}$$

Quite similarly, using the corresponding quantifier elimination command for the upper bounds r_{2b}

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, r ≤ r2b, exp[x] == e0[x]]], Reals]
        //Timing                                                           [c7]
```

again in just about 0.5 s we find the related simple QFF (quantifier-free formula)

$$r_{2b} \geq 0.0129428. \tag{26}$$

Obviously, the conjunction of these two QFFs (25) and (26) simply constitutes the original QFF (21), but the reduction of the CPU time (computational time) is very significant: about $0.5\ \text{s} + 0.5\ \text{s} = 1\ \text{s}$ now (with only one free variable, either r_{1b} or r_{2b}, in the quantifier elimination commands [c6] or [c7], respectively) instead of about 60 s previously (with two free variables, r_{1b} and r_{2b}, simultaneously in the initial quantifier elimination command [c5]).

Additionally, clearly, we can also combine the previous quantifier elimination commands [c6] and [c7] to the single composite quantifier elimination command with the conjunction operator ∧

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, r ≥ r1b, exp[x] == e0[x]]], Reals] ∧
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, r ≤ r2b, exp[x] == e0[x]]], Reals]
        //Chop//Timing                                                    [c8]
```

Then we directly obtain the QFF (21) in about 1 s (much less CPU time than 60 s) as is expected.

But, on the other hand, alternatively and preferably, we can also use the purely existentially quantified formula

$$\exists x \in X = [1,2] \ \text{ such that } \ \exp_{25} x = e_0(x) \tag{27}$$

(but here with only one quantified variable, the variable x) instead of the original quantified formula (20) (with two quantified variables: the variables x and r), where the variable $r \in R$ in $e_0(x)$ is now a free variable. The related and now simpler quantifier elimination command has the form

```
Reduce[Exists[x, 1 ≤ x ≤ 2, exp[x] == e0[x]], Reals]//Chop//Timing        [c9]
```

with resulting QFF (quantifier-free formula) the very simple and surely preferable QFF

$$0 \leq r \leq 0.0129428, \tag{28}$$

which is, clearly, simpler than the QFF (21). Moreover, the above QFF (28) has been computed in just about 0.6 s instead of about 60 s having been required previously for the more complicated QFF (21) and, naturally, it directly leads to the same result (22) for the interval R in the interval-based cubic polynomial $e_0(x)$ in Eq. (15). Therefore, the approach based on the purely existentially quantified formula (27) is the preferable one and, hence, we will continuously use it from now on.

Evidently, quite analogously, we can work for the second interval-based cubic polynomial $e_1(r)$ in Eq. (16). The related quantified formula is again the existentially quantified formula (27), but now, naturally, with $e_1(x)$ instead of $e_0(x)$, and the related quantifier elimination command is again the above command [c9], but now, analogously, with `e1[x]` instead of `e0[x]`. The resulting QFF (quantifier-free formula) has again the very simple form

$$0 \leq r \leq 0.0105947 \tag{29}$$

and it was computed in about 0.9 s. Therefore, the interval R (with $r \in R$) in $e_1(x)$ in Eq. (16) is the interval

$$R = [0, 0.0105947]. \tag{30}$$

In a completely analogous manner, we can also work with the three next interval-based cubic polynomials $e_2(x)$, $e_3(x)$ and $e_r(x)$ in Eqs. (17), (18) and (19), respectively. The resulting intervals R (with $r \in R$), which were also computed with the method of quantifier elimination, are the same interval R in Eq. (30) as far as the two interval-based cubic polynomials $e_2(x)$ and $e_3(x)$ in Eqs. (17) and (18), respectively, are concerned (with $r \in R$) and, finally, the different interval

$$R = [0, 0.00391282] \tag{31}$$

as far as the last interval-based cubic polynomial $e_r(x)$ in Eq. (19) is concerned (again with $r \in R$).

3.3. A simple application to quantifier elimination for an upper bound of the exponential function

As a simple application of the present generalized interval-based polynomial approximations to the exponential function $e^x \equiv \exp x$ we now consider the special case where we wish that this function does not exceed the upper bound $c = 7.38$ in the whole interval $X = [1, 2]$ under consideration (i.e. with $x \in X = [1, 2]$). Therefore, here we have the very simple universally quantified formula

$$\forall x \in X = [1, 2] \text{ it holds true that } \exp x \leq c. \tag{32}$$

Naturally, exactly as previously, we use again the approximation $\exp_{25} x$ in Eq. (12) to $\exp x$ instead of $\exp x$ with the error being of the order of 10^{-15} as is clear from Eqs. (13) and, therefore, negligible with respect to the present computations. We denote again the function $\exp_{25} x \approx \exp x$ by the symbol `exp[x]` in *Mathematica* and, next, the function $\exp_3 x$ in Eq. (14) (an approximation to $\exp x$ by a cubic polynomial) by the symbol `exp3[x]`. Moreover, we denote the interval-based cubic polynomial approximations $e_0(x)$, $e_1(x)$, $e_2(x)$, $e_3(x)$ and $e_r(x)$ in Eqs. (15)–(19) by the related symbols `e0[x]`, `e1[x]`, `e2[x]`, `e3[x]` and `er[x]`, respectively.

At first, we use the quantifier elimination commands

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, exp[x] ≤ c]]                                   [c10]
Reduce[ForAll[x, 1 ≤ x ≤ 2, exp3[x] ≤ c]]                                  [c11]
```

with resulting QFFs (quantifier-free formulae) simply `False` and `True`, respectively. The first result, `False`, is reasonable since, as is well known, the exponential function $\exp x$ is a monotonically increasing function with $\exp 2 \approx 7.38906 > c = 7.38$. The second result, `True`, is also reasonable since $\exp_3 2 \approx 7.37611 < c = 7.38$. But, naturally, if we wish to use the cubic polynomial approximation $\exp_3 x$ in Eq. (14) to $\exp x$, we also wish to obtain correct results as much as possible and, in our case, the correct result `False` already obtained by using the first quantifier elimination command [c10], but, unfortunately, not obtained with the second quantifier elimination command [c11]. The latter command provided us with the incorrect result `True`. Evidently, this happened simply because of the use of the approximation $\exp_3 x$ in Eq. (14) to the exponential function $\exp x$.

In order to obtain the correct result, `False`, we can appropriately use the interval-based cubic polynomial approximation $e_0(x)$ or $e_1(x)$ or $e_2(x)$ or $e_3(x)$ or $e_r(x)$ in Eqs. (15)–(19) by using one of the following five quantifier elimination commands:

```
Reduce[ForAll[{x,r}, 1 ≤ x ≤ 2 ∧ qff0[[2]], e0[x] ≤ c], Reals]             [c12]
Reduce[ForAll[{x,r}, 1 ≤ x ≤ 2 ∧ qff1[[2]], e1[x] ≤ c], Reals]             [c13]
Reduce[ForAll[{x,r}, 1 ≤ x ≤ 2 ∧ qff2[[2]], e2[x] ≤ c], Reals]             [c14]
Reduce[ForAll[{x,r}, 1 ≤ x ≤ 2 ∧ qff3[[2]], e3[x] ≤ c], Reals]             [c15]
Reduce[ForAll[{x,r}, 1 ≤ x ≤ 2 ∧ qffr[[2]], er[x] ≤ c], Reals]             [c16]
```

respectively, where `qff0[[2]]`, `qff1[[2]]`, `qff2[[2]]`, `qff3[[2]]` and `qffr[[2]]` denote the five related QFFs (quantifier-free formulae), i.e. the corresponding five inequality constraints of the form $0 \leq r \leq r_0$ (essentially intervals of the form $R = [0, r_0]$) having been obtained in the previous subsection. The results of all the above five commands were simply `False` as was already mentioned. This has been simply due to the fact that now we have considered the variable r (lying in the interval R, i.e. with $r \in R$) in Eqs. (15)–(19) to be universally quantified and this quantification includes the worst case for r. On the contrary, by using the corresponding universally–existentially quantifier elimination commands with the same variable r now existentially quantified (with this quantification obviously including the most favourable case), i.e. the following five quantifier elimination commands:

```
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, qff0[[2]], e0[x] ≤ c]], Reals]      [c17]
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, qff1[[2]], e1[x] ≤ c]], Reals]      [c18]
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, qff2[[2]], e2[x] ≤ c]], Reals]      [c19]
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, qff3[[2]], e3[x] ≤ c]], Reals]      [c20]
Reduce[ForAll[x, 1 ≤ x ≤ 2, Exists[r, qffr[[2]], er[x] ≤ c]], Reals]      [c21]
```

we obtain the simple QFF (quantifier-free formula) `True` in all five cases, i.e. an incorrect QFF in the present elementary application concerning the exponential function. This happens simply because now we have restricted our attention to the existence of just one value of r in the related interval $R = [0, r_0]$ in the interval-based cubic polynomial approximations (15)–(19).

4. A beam on a Winkler elastic foundation

4.1. The problem of a beam on a Winkler elastic foundation and the deflection of the beam

In this section, we consider the problem of an ordinary beam on a Winkler elastic foundation of modulus k_0 [105, p. 2]. (The monograph by Hetényi [105] is still the classical monograph on beams on elastic foundation.) More explicitly, exactly as in Ref. [18, Section 4, pp. 11–13], we consider a cantilever beam of rectangular cross-section of width b, length L and flexural rigidity EI clamped (fixed) at its left end $x = 0$ and free at its right end $x = L$. The beam is assumed loaded by a compressive concentrated normal load P at its free end $x = L$ with $x \in [0, L]$ [105, p. 64]. For convenience here we use the dimensionless variable $\xi := x/L$ with $\xi \in [0, 1]$ on the whole beam.

Under these conditions the deflection $y(\xi)$ of the beam has the form $y(\xi) = Q v_\mu(\xi)$ [105, p. 64], where Q is a positive overall parameter (constant) and the function $v_\mu(\xi)$ is an appropriate dimensionless transcendental function, which is directly seen to be given by [105, p. 64, Eq. (51a)]

$$v_\mu(\xi) = \cosh\mu \sinh(\mu\xi)\cos[\mu(1-\xi)] - \cos\mu \sin(\mu\xi)\cosh[\mu(1-\xi)] \quad \text{with} \quad \xi \in [0, 1]. \tag{33}$$

This equation was written here in a more convenient form exactly as in Ref. [18, p. 11, Eq. (49)]. In the above equation, μ is a dimensionless positive overall parameter (constant) of the present mechanical system, beam–Winkler elastic foundation, defined by [18, p. 11, Eqs. (50)]

$$\mu := \lambda L, \tag{34}$$

where the parameter (constant) λ is defined by

$$\lambda := \sqrt[4]{\frac{bk_0}{4EI}}. \tag{35}$$

Here we restrict our attention to the very simple case where the overall parameter (constant) μ of the present mechanical system beam–Winkler elastic foundation takes the simple value $\mu = 1$. In this particular case, Eq. (33) takes the following simpler form (with $v(\xi) := v_\mu(x)$ for $\mu = 1$):

$$v(\xi) = \cosh 1 \sinh\xi \cos(1-\xi) - \cos 1 \sin\xi \cosh(1-\xi) \quad \text{again with} \quad \xi \in [0, 1]. \tag{36}$$

Since the method of quantifier elimination cannot generally be used with transcendental functions as has been already mentioned, here we will use a high-accuracy Taylor–Maclaurin series approximation to the above function $v(\xi)$ in Eq. (36). This polynomial approximation $v_{20}(\xi) \approx v(\xi)$ is directly obtained by using the `Series` command of *Mathematica* [2] more explicitly the following command (here with terms up to $n = 20$ in the resulting polynomial approximation):

```
v20[xi_] = N[Series[v[xi], {xi,0,20}]//Normal, 20]      [c22]
```

and it has the following form (here with an accuracy of twenty significant digits):

$$\begin{aligned}v_{20}(\xi) = {} & -7.0182814011004027641\cdot 10^{-15}\xi^{19} + 1.5461637379416366672\cdot 10^{-13}\xi^{18}\\ & + 1.6321715226399096668\cdot 10^{-10}\xi^{15} - 2.8387566228608449209\cdot 10^{-9}\xi^{14}\\ & - 1.3367484770420860171\cdot 10^{-6}\xi^{11} + 0.000017049572276902234595\xi^{10}\\ & + 0.0026467619845433303139\xi^{7} - 0.021482461068896815590\xi^{6}\\ & - 0.55582001675409936592\xi^{3} + 1.9334214962007134031\xi^{2}. \end{aligned} \tag{37}$$

In the above Taylor–Maclaurin series approximation $v_{20}(\xi)$ to the dimensionless deflection $v(\xi)$ of the beam, we observe that the terms corresponding to several powers of ξ including ξ^0, ξ^1 and ξ^{20} are missing. Of course, this is completely natural and justified for the powers ξ^0 and ξ^1 because the present beam has been already assumed clamped (fixed) at its left end $x = 0$.

Moreover, we directly find with the help of *Mathematica* [2] (by using both its `NMinimize` and `NMaximize` commands on the interval $\xi \in [0,1]$) that the error $d_v(\xi) := v(\xi) - v_{20}(\xi)$ of the above approximation $v_{20}(\xi)$ in Eq. (37) lies in the interval $[-8.88178, 2.22045]\cdot 10^{-16}$. Therefore, this error is negligible in the present computations, where we intend to use the Taylor–Maclaurin series approximation $v_{20}(\xi) \approx v(\xi)$ instead of $v(\xi)$ itself during the quantifier eliminations to be performed below by using the quantifier elimination implementation available in *Mathematica*.

Here we will also employ the following Taylor–Maclaurin series approximations to $v(\xi)$:

$$v_2(\xi) = +1.9334214962007134031\xi^2, \tag{38}$$

$$v_3(\xi) = -0.55582001675409936592\xi^3 + 1.9334214962007134031\xi^2, \tag{39}$$

$$\begin{aligned}v_6(\xi) = {} & -0.021482461068896815590\xi^6 - 0.55582001675409936592\xi^3\\ & + 1.9334214962007134031\xi^2\end{aligned} \tag{40}$$

having been directly obtained again by using the `Series` command of *Mathematica* and, obviously, consisting of the last terms of $v_{20}(\xi)$ in Eq. (37) as is directly observed. Moreover, since the three powers ξ^1, ξ^4 and ξ^5 of ξ do not appear in $v_6(\xi)$ in Eq. (40), evidently, it is meaningless to use the related functions $v_1(\xi)$, $v_4(\xi)$ and $v_5(\xi)$.

Now, on the basis of the above polynomial approximations and, especially, $v_6(\xi)$, here we will use the following four interval-based sixth-degree polynomial approximations to the dimensionless deflection $v(\xi)$ of the beam:

$$w_0(\xi) = v_6(\xi) + r, \tag{41}$$

$$w_2(\xi) = v_6(\xi) + r v_2(\xi), \tag{42}$$

$$w_3(\xi) = v_6(\xi) + r v_3(\xi), \tag{43}$$

$$w_r(\xi) = (1 + r) v_6(\xi). \tag{44}$$

In the above four polynomial approximations $w_0(\xi)$, $w_2(\xi)$, $w_3(\xi)$ and $w_r(\xi)$, the symbol r denotes a parameter lying in an appropriate interval R (clearly different for each polynomial approximation) to be determined below (i.e. $r \in R$). These intervals R will be computed below by using the method of quantifier elimination in *Mathematica* [2] exactly as has been already the case in the previous section, Section 3, for the exponential function $e^x \equiv \exp x$ studied there.

4.2. Computation of the intervals in the generalized interval-based polynomial approximations

For the computation of the intervals R (with $r \in R$) in the above four interval-based sixth-degree polynomial approximations $w_0(\xi)$, $w_2(\xi)$, $w_3(\xi)$ and $w_r(\xi)$ in Eqs. (41), (42), (43) and (44), respectively, to the dimensionless deflection $v(\xi)$ on the whole beam (with $\xi \in [0,1]$) here using its high-accuracy approximation $v_{20}(\xi)$ in Eq. (37), we can use the existentially quantified formulae

$$\exists \xi \in [0,1] \text{ such that } v_{20}(\xi) = w_0(\xi), \tag{45}$$

$$\exists \xi \in [0,1] \text{ such that } v_{20}(\xi) = w_2(\xi), \tag{46}$$

$$\exists \xi \in [0,1] \text{ such that } v_{20}(\xi) = w_3(\xi), \tag{47}$$

$$\exists \xi \in [0,1] \text{ such that } v_{20}(\xi) = w_r(\xi). \tag{48}$$

The related quantifier elimination commands used in *Mathematica* [2] have the forms

```
qff0 = Reduce[Exists[xi, 0 ≤ xi ≤ 1, v20[xi]/xi^2 == w0[xi]/xi^2], Reals]//Timing [c23]
qff2 = Reduce[Exists[xi, 0 ≤ xi ≤ 1, v20[xi]/xi^2 == w2[xi]/xi^2], Reals]//Timing [c24]
qff3 = Reduce[Exists[xi, 0 ≤ xi ≤ 1, v20[xi]/xi^2 == w3[xi]/xi^2], Reals]//Timing [c25]
qffr = Reduce[Exists[xi, 0 ≤ xi ≤ 1, v20[xi]/xi^2 == w6[xi]/xi^2], Reals]//Timing [c26]
```

where we have divided by ξ^2 in order to avoid the zeros at the point $\xi = 0$, i.e. the left end (the clamped end) of the present beam on a Winkler elastic foundation with $\xi \in [0,1]$. The resulting QFFs (quantifier-free formulae), i.e. essentially the resulting intervals R (with $r \in R$), have the forms

$$0 \le r \le 0.0026624721329513179580, \quad \text{i.e.} \quad R = [0, 0.0026624721329513179580], \tag{49}$$

$$0 \le r \le 0.0013770779616256630024, \quad \text{i.e.} \quad R = [0, 0.0013770779616256630024], \tag{50}$$

$$0 \le r \le 0.0019326867549683814791, \quad \text{i.e.} \quad R = [0, 0.0019326867549683814791], \tag{51}$$

$$0 \le r \le 0.0019633027019533655094, \quad \text{i.e.} \quad R = [0, 0.0019633027019533655094]. \tag{52}$$

These four QFFs correspond to the interval-based approximations $w_0(\xi)$, $w_2(\xi)$, $w_3(\xi)$ and $w_r(\xi)$ to the dimensionless deflection $v(\xi)$ of the beam in Eqs. (41), (42), (43) and (44), respectively.

4.3. An application to quantifier elimination for the upper bound of the deflection of the beam

As a simple application of the present interval-based polynomial approximations to the dimensionless deflection $v(\xi)$ of the present beam on a Winkler elastic foundation we consider the special case where we wish that this function $v(\xi)$ does not exceed the upper bound $h = 1.357$ on the whole beam under consideration with $\xi \in [0,1]$. Hence, we have the simple universally quantified formula

$$\forall \xi \in [0,1] \text{ it holds true that } v(\xi) \le h. \tag{53}$$

Naturally, exactly as previously, we generally employ again the approximation $v_{20}(\xi)$ in Eq. (37) to $v(\xi)$ instead of $v(\xi)$ itself with the error being of the order of 10^{-15} to 10^{-16} as has been already mentioned and, therefore, negligible as far as the present computations are concerned. Additionally, we use again the four interval-based sixth-degree polynomial approximations $w_0(\xi)$, $w_2(\xi)$, $w_3(\xi)$ and $w_r(\xi)$ in Eqs. (41), (42), (43) and (44), respectively, exactly as previously.

At first, by using the maximization command

```
{Maximize[v[xi], 0 ≤ xi ≤ 1, xi],  Maximize[v6[xi], 0 ≤ xi ≤ 1, xi]}//N        [c27]
```

(here having been able to use $v(\xi)$ itself instead of its approximation $v_{20}(\xi)$), we observe that

$$v_{\max} = 1.35878 > h \quad \text{whereas} \quad v_{6,\max} = 1.35612 < h \quad \text{in both cases for } \xi = 1. \tag{54}$$

Therefore, it is completely reasonable that the two quantifier elimination commands

```
Reduce[ForAll[xi, 0 ≤ xi ≤ 1, v20[xi] ≤ h], Reals]        [c28]
Reduce[ForAll[xi, 0 ≤ xi ≤ 1, v6[xi] ≤ h], Reals]         [c29]
```

provide us with completely different results (here QFFs): `False` and `True`, respectively.

But, clearly, because we wish to mainly use the sixth-degree polynomial approximation $v_6(\xi)$ to $v(\xi)$, we also wish to get correct results as much as possible. In our case, the correct result is `False` and it was already obtained by using the first quantifier elimination command [c28], but not the second quantifier elimination command [c29]. The latter provided us with the incorrect result `True` simply because of the use of the approximation $v_6(\xi)$ to the dimensionless deflection $v(\xi)$ of the beam, which is not a sufficiently good approximation in the present particular application.

In order to obtain the correct result, `False`, here we can appropriately use the interval-based sixth-degree polynomial approximation $w_0(\xi)$ or $w_2(\xi)$ or $w_3(\xi)$ or $w_r(\xi)$ in Eqs. (41), (42), (43) and (44), respectively. This is rather easily achieved by using the quantifier elimination commands

```
Reduce[ForAll[{xi,r}, 0 ≤ xi ≤ 1 ∧ qff0[[2]], w0[xi] ≤ h], Reals]      [c30]
Reduce[ForAll[{xi,r}, 0 ≤ xi ≤ 1 ∧ qff2[[2]], w2[xi] ≤ h], Reals]      [c31]
Reduce[ForAll[{xi,r}, 0 ≤ xi ≤ 1 ∧ qff3[[2]], w3[xi] ≤ h], Reals]      [c32]
Reduce[ForAll[{xi,r}, 0 ≤ xi ≤ 1 ∧ qffr[[2]], wr[xi] ≤ h], Reals]      [c33]
```

respectively, with resulting QFF the simple (and correct) QFF `False` in all four these commands.

We should add that in the above commands the symbols `qff0[[2]]`, `qff2[[2]]`, `qff3[[2]]` and `qffr[[2]]` denote the related QFFs (quantifier-free formulae), i.e. the corresponding inequality constraints of the form $0 \le r \le r_0$ (essentially intervals of the form $R = [0, r_0]$ with $r \in R$) having been obtained in the previous subsection, Eqs. (49), (50), (51) and (52), respectively. The result of all the above four commands has been simply `False` as it should be and was already mentioned. This correct result was simply due to the fact that here we have considered the interval variable r in Eqs. (41), (42), (43) and (44) universally quantified and this quantification includes the related worst case for r on the beam thus permitting us to obtain the correct result here the result `False`.

A related application concerns the more general case where the upper bound h of the dimensionless deflection $v(\xi)$ of the present beam on a Winkler elastic foundation is simply a symbol and not a number, e.g. $h = 1.357$ previously. In this case, having previously used the simple command

```
Clear[h]      [c34]
```

we have again the quantified formula (53) (but now with the parameter h in it being simply a symbol) and the related quantifier elimination command [c28]. The resulting QFF has the simple form

$$h_e \ge 1.3587814905106685395. \tag{55}$$

Alternatively, by using the sixth-degree polynomial approximation $v_6(\xi)$ in Eq. (40) to $v(\xi)$ in Eq. (36) and the related quantifier elimination command [c29] we obtain the approximate bound

$$h_6 \ge 1.3561190183777172216, \tag{56}$$

which is somewhat smaller than the almost exact bound (55) and, therefore, not so reliable.

Now, in order to be able to get reliable bounds for h, naturally, we can use one of the four sixth-degree interval-based polynomial approximations $w_0(\xi)$ or $w_2(\xi)$ or $w_3(\xi)$ or $w_r(\xi)$ displayed in Eqs. (41), (42), (43) and (44), respectively, exactly as previously, but with a numerical value of h, i.e. $h = 1.357$, there. To this end, naturally, we have employed again the four quantifier elimination commands [c30], [c31], [c32] and [c33], but now with h being a symbolic parameter. In all these four cases, the resulting bound for h has been the bound h_e in Eq. (55), i.e., essentially, the exact bound. (Of course, this happens approximately because of the approximations already made in the interval-based polynomials $w_0(\xi)$ or $w_2(\xi)$ or $w_3(\xi)$ or $w_r(\xi)$.) This result illustrates the practical usefulness of the present generalized interval-based polynomial approximations (here sixth-degree polynomial approximations to a rather complicated transcendental function) in applied mechanics.

5. Free vibrations of a harmonic oscillator with critical damping

5.1. The displacement of the harmonic oscillator with critical damping

In this section, we consider the classical applied-mechanics problem of free vibrations of a harmonic oscillator with critical damping. The displacement of the oscillator has the very well-known form

$$u(\tau) = c_1 e^{-\tau}(1 + c\tau) \quad \text{with} \quad \tau := \omega_0 t, \tag{57}$$

where t denotes time (with $t, \tau \geq 0$), ω_0 the natural frequency of vibrations of the related oscillator without damping and c_1 and c are two constants which depend on the initial conditions [18, p. 18, Eq. (88)]. Here we assume that $c_1 = 1$ (dimensionless displacement $u(\tau)$) and $c = 2$. Then we have

$$u(\tau) = e^{-\tau}(1 + 2\tau). \tag{58}$$

This is the function that we wish to approximate here by using generalized interval-based polynomial approximations and the method of quantifier elimination with the help of *Mathematica* [2].

Of course, we understand again that quantifier elimination is generally inapplicable to transcendental functions such as the exponential function in the right-hand side of Eq. (58). Therefore, we approximate the displacement $u(\tau)$ in Eq. (58) by using a Taylor–Maclaurin series approximation with terms up to τ^{30} analogously to what we have done in the previous two sections, Section 3 and Section 4, for the two transcendental functions studied there. This series is easily computed by using the `Series` command of *Mathematica* (together with the `Normal` and `N` commands) as follows:

```
u30[τ_] = N[Series[u[τ], {τ,0,30}]//Normal, 20]                [c35]
```

The resulting series (polynomial of the thirtieth degree in the dimensionless variable τ) has the form

$$\begin{aligned} u_{30}(\tau) = {} & -2.2242927010013843299 \cdot 10^{-31}\tau^{30} + 6.4466788452751986510 \cdot 10^{-30}\tau^{29} \\ & - 1.8039390803884108506 \cdot 10^{-28}\tau^{28} + 4.8673556278116394587 \cdot 10^{-27}\tau^{27} \\ & - 1.2645940942446467046 \cdot 10^{-25}\tau^{26} + 3.1590056393483919641 \cdot 10^{-24}\tau^{25} \\ & - 7.5751665841517562405 \cdot 10^{-23}\tau^{24} + 1.7406765767838078170 \cdot 10^{-21}\tau^{23} \\ & - 3.8256202987537465133 \cdot 10^{-20}\tau^{22} + 8.0249058359904171046 \cdot 10^{-19}\tau^{21} \\ & - 1.6030238730917442948 \cdot 10^{-17}\tau^{20} + 3.0416350412510019953 \cdot 10^{-16}\tau^{19} \\ & - 5.4667224390051792618 \cdot 10^{-15}\tau^{18} + 9.2778089393402185186 \cdot 10^{-14}\tau^{17} \\ & - 1.4816379730400894422 \cdot 10^{-12}\tau^{16} + 2.2176774822277467780 \cdot 10^{-11}\tau^{15} \\ & - 3.0971013113870256727 \cdot 10^{-10}\tau^{14} + 4.0147609592054036498 \cdot 10^{-9}\tau^{13} \\ & - 4.8016541072096627652 \cdot 10^{-8}\tau^{12} + 5.2609427609427609428 \cdot 10^{-7}\tau^{11} \\ & - 5.2358906525573192240 \cdot 10^{-6}\tau^{10} + 0.000046847442680776014109\tau^{9} \\ & - 0.00037202380952380952381\tau^{8} + 0.0025793650793650793651\tau^{7} \\ & - 0.015277777777777777778\tau^{6} + 0.075000000000000000000\tau^{5} \\ & - 0.29166666666666666667\tau^{4} + 0.83333333333333333333\tau^{3} \\ & - 1.5\tau^{2} + \tau + 1. \end{aligned} \tag{59}$$

The above series approximation, $u_{30}(\tau) \approx u(\tau)$, was seen to be of sufficient accuracy for our purposes. In fact, we easily find using the `NMinimize` and `NMaximize` commands of *Mathematica* (exactly as in the previous section, Section 4) that in the interval $\tau \in [0, 5]$, where we are interested here, the difference $d_u(\tau) := u(\tau) - u_{30}(\tau)$ lies in the interval $[-8.18789 \cdot 10^{-15}, 2.97978 \cdot 10^{-11}]$.

Of course, other types of approximation to $u(\tau)$ could alternatively have been used here instead of the Taylor–Maclaurin series approximation $u_{30}(\tau)$ in Eq. (59) such as minimax approximations. Minimax approximations will be actually used just below for the computation of interval-based polynomial approximations to $u(\tau)$ of interest here. We also note that minimax approximations have also been used in the present problem in Ref. [18, Section 6, pp. 18–21], but only for the computation of the range of the displacement $u(\tau)$ of the oscillator with critical damping there.

5.2. Minimax polynomial approximations to the displacement of the harmonic oscillator

As has been already mentioned, our present interval-based polynomial approximations to the displacement $u(\tau)$ of the harmonic oscillator in Eq. (58) will be based on minimax approximations to this function. For the computation of these approximations we can simply use *Mathematica* [2] by loading its function approximations package by using the command

```
<<FunctionApproximations`                                   [c36]
```

The *Mathematica* function of interest here is the function `MiniMaxApproximation` of this package. By using the following commands based on this function for minimax polynomial approximations:

```
minimax2 = MiniMaxApproximation[u[τ],{τ,{0,5},2,0}]         [c37]
minimax3 = MiniMaxApproximation[u[τ],{τ,{0,5},3,0}]         [c38]
minimax4 = MiniMaxApproximation[u[τ],{τ,{0,5},4,0}]         [c39]
minimax5 = MiniMaxApproximation[u[τ],{τ,{0,5},5,0}]         [c40]
minimax6 = MiniMaxApproximation[u[τ],{τ,{0,5},6,0}]         [c41]
```

we easily find the following polynomial minimax approximations to $u(\tau)$:

$$u_2(\tau) = +0.0224909\tau^2 - 0.340712\tau + 1.20054, \tag{60}$$

$$u_3(\tau) = +0.015222\tau^3 - 0.10368\tau^2 - 0.0716583\tau + 1.1314, \tag{61}$$

$$u_4(\tau) = -0.00982706\tau^4 + 0.125951\tau^3 - 0.507012\tau^2 + 0.418264\tau + 1.05221, \tag{62}$$

$$u_5(\tau) = +0.00321333\tau^5 - 0.0545465\tau^4 + 0.348883\tau^3 - 0.974333\tau^2 + 0.771595\tau + 1.01506, \tag{63}$$

$$u_6(\tau) = -0.000748162\tau^6 + 0.0155531\tau^5 - 0.132117\tau^4 + 0.578159\tau^3 - 1.28772\tau^2 + 0.930667\tau + 1.00345 \tag{64}$$

of degrees 2, 3, 4, 5 and 6, respectively. Here we assume that $u_6(\tau)$ is our fundamental approximation to $u(\tau)$ in Eq. (58) and, as was already mentioned, we will continuously use $u_{30}(\tau)$ in Eq. (59) instead of $u(\tau)$ because the exponential function in Eq. (58) for $u(\tau)$ is generally unacceptable during quantifier elimination, which is applicable mainly to polynomial and to rational functions.

5.3. The generalized interval-based polynomial approximations to the displacement of the oscillator

On the basis of the above five minimax polynomial approximations (60)–(64) to the displacement $u(\tau)$ of the present oscillator with critical damping, we now construct interval-based sixth-degree polynomial approximations to this displacement, all of which have the sixth-degree minimax polynomial approximation $u_6(\tau)$ in Eq. (64) to $u(\tau)$ in Eq. (58) as their fundamental approximation.

At first, we assume the interval-based simple polynomial approximation

$$u_{r0}(\tau) = u_6(\tau) + r, \tag{65}$$

where r is an interval parameter belonging to an interval R (i.e. $r \in R$) to be determined. For the determination of this interval R (the range of the interval parameter r with $r \in R$) we will use again the method of quantifier elimination, which is performed to the existentially quantified formula

$$\exists \tau \in [0,5] \text{ such that } u_{30}(\tau) = u_{r0}(\tau). \tag{66}$$

Here, as has been already mentioned, we have used the high-accuracy approximation $u_{30}(\tau)$ to the displacement $u(\tau)$. The related quantifier elimination command in *Mathematica* [2] has the form

```
qff0 = Reduce[Exists[τ, 0 ≤ τ ≤ 5, u30[τ] == ur0[τ]], Reals]          [c42]
```

and the resulting QFF (quantifier-free formula) for $u_{r0}(\tau)$, i.e. for $u_{rk}(\tau)$ for $k = 0$, is

$$-0.00368836 \le r \le 0.00404789, \quad \text{i.e.} \quad R = [-0.00368836, 0.00404789] \text{ for } k = 0. \tag{67}$$

Next, we consider the generalized interval-based polynomial approximation

$$u_{r2}(\tau) = u_6(\tau) + r u_2(\tau), \tag{68}$$

where both minimax approximations $u_6(\tau)$ and $u_2(\tau)$ in Eqs. (64) and (60), respectively, are used. For the determination of the interval R (the range of the interval-valued parameter r with $r \in R$) we can use again the method of quantifier elimination performed to the existentially quantified formula

$$\exists \tau \in [0,5] \text{ such that } u_{30}(\tau) = u_{r2}(\tau). \tag{69}$$

The related quantifier elimination command in *Mathematica* [2] has the form

```
qff2 = Reduce[Exists[τ, 0 ≤ τ ≤ 5, u30[τ] == ur2[τ]], Reals]          [c43]
```

(which is similar to the previous command [c42]) and the resulting QFF (quantifier-free formula) is

$$-0.00431011 \le r \le 0.00431921, \quad \text{i.e.} \quad R = [-0.00431011, 0.00431921] \text{ for } k = 2. \tag{70}$$

Completely analogously, we can work with the three generalized interval-based approximations

$$u_{rk}(\tau) = u_6(\tau) + r u_k(\tau) \quad \text{with } k = 3, 4 \text{ and } 5, \tag{71}$$

where the minimax approximations $u_k(\tau)$ (with $k = 3, 4, 5$ and 6) to the displacement $u(\tau)$ of the oscillator are given by Eqs. (61), (62), (63) and (64), respectively. The resulting QFFs (quantifier-free formulae) and the related intervals R (the ranges of the interval-valued parameter r) are

$$-0.00387869 \le r \le 0.00387631, \quad \text{i.e.} \quad R = [-0.00387869, 0.00387631] \text{ for } k = 3, \tag{72}$$

$$-0.00360707 \le r \le 0.00364327, \quad \text{i.e.} \quad R = [-0.00360707, 0.00364327] \text{ for } k = 4, \tag{73}$$

$$-0.00350175 \le r \le 0.00350136, \quad \text{i.e.} \quad R = [-0.00350175, 0.00350136] \text{ for } k = 5. \tag{74}$$

The above intervals R (the ranges of r with $r \in R$) assure us that $\forall \tau \in [0,5]$ the actual values of the displacement $u(\tau)$ of the present oscillator with critical damping belong to the generalized sixth-degree interval-based polynomial approximations $u_{rk}(\tau)$ (with $k = 0, 2, 3, 4, 5$) for appropriate values of the interval-valued parameter r belonging to the corresponding interval R ($r \in R$), i.e.

$$\forall \tau \in [0,5] \text{ it holds true that } u(\tau) \in [u_{rk}(\tau)|_{r=r_1}, u_{rk}(\tau)|_{r=r_2}] \quad \text{here with } k = 0, 2, 3, 4 \text{ and } 5. \tag{75}$$

Here r_1 and r_2 denote the two corresponding endpoints of the above intervals $R = [r_1, r_2]$ of the parameter r. (These five intervals R are different for each of the functions $u_{rk}(\tau)$ with $k = 0, 2, 3, 4$ and 5 as is clear from Eqs. (67), (70), (72), (73) and (74), respectively.) In Fig. 2 (on the next page), we display the differences $\Delta u_{k,1} := u(\tau) - u_{rk}(\tau)|_{r=r_1}$ (with positive values, curves at the upper half-planes) and $\Delta u_{k,2} := u(\tau) - u_{rk}(\tau)|_{r=r_2}$ (with negative values, curves at the lower half-planes) for the two endpoints $r = r_1$ and $r = r_2$ of the intervals R of r, respectively, with $k = 0, 2, 3, 4$ and 5. We observe that the related graphical representations in Fig. 2 (with $k = 0, 2, 3, 4$ and 5) are similar.

Finally, we can mention that the use of the generalized interval-based polynomial approximation $u_{rr}(\tau) = (1 + r)u_6(\tau)$, which is based only on $u_6(\tau)$, is also possible although it requires too much computational time. The resulting interval was found to be $R = [-0.00344116, 0.00346500]$.

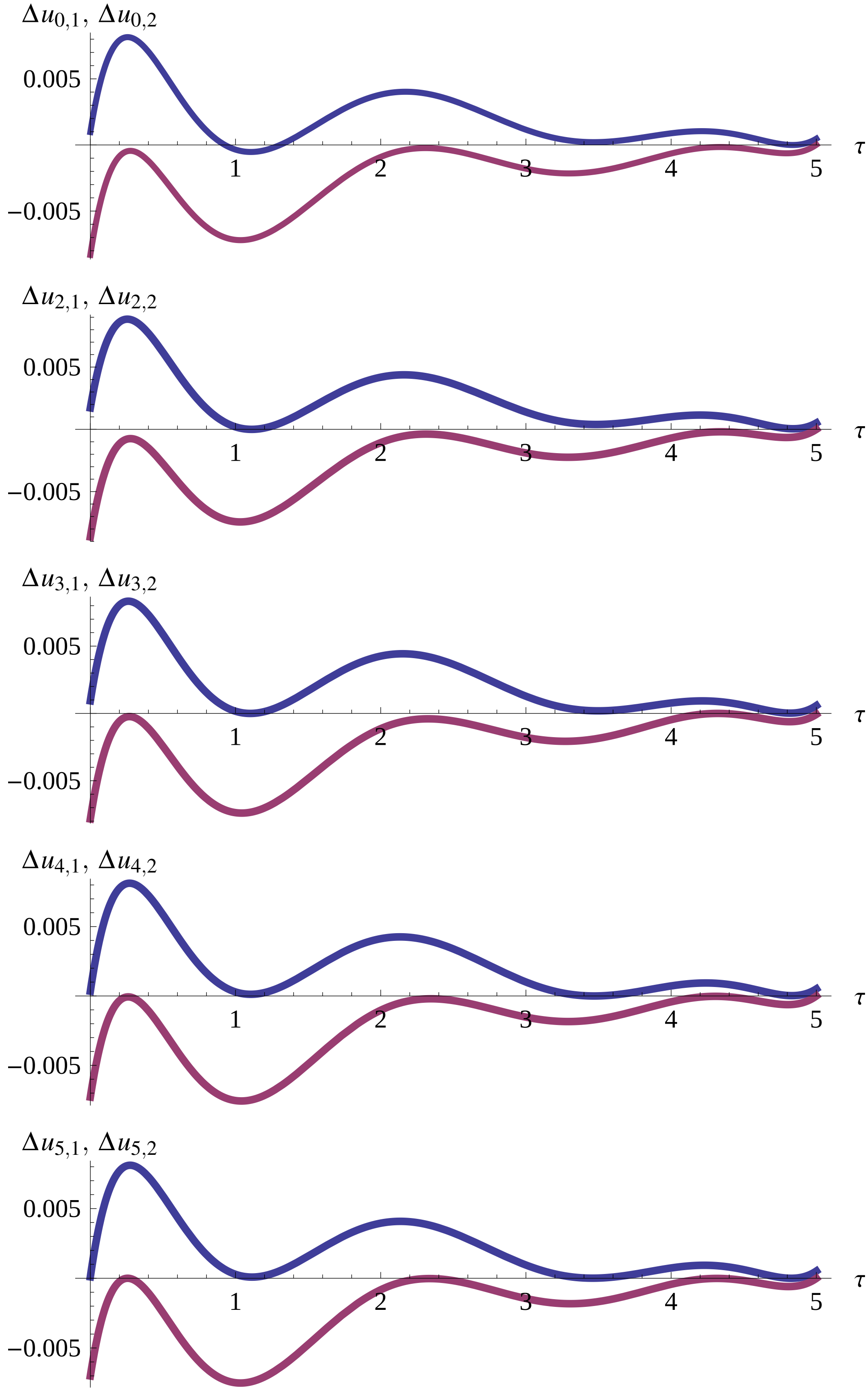

Fig. 2. The differences $\Delta u_{k,1} := u(\tau) - u_{rk}(\tau)|_{r=r_1}$ (with positive values, curves at the upper half-planes) and $\Delta u_{k,2} := u(\tau) - u_{rk}(\tau)|_{r=r_2}$ (with negative values, curves at the lower half-planes) for the endpoints $r = r_1$ and $r = r_2$ of the intervals R of r, respectively, and for $k = 0, 2, 3, 4$ and 5 with the displacement $u(\tau)$ of the oscillator given by Eq. (58) and the interval-valued functions $u_{rk}(\tau)$ (approximations to $u(\tau)$) defined by Eqs. (65) for $k = 0$, (68) for $k = 2$ and (71) for $k = 3, 4$ and 5.

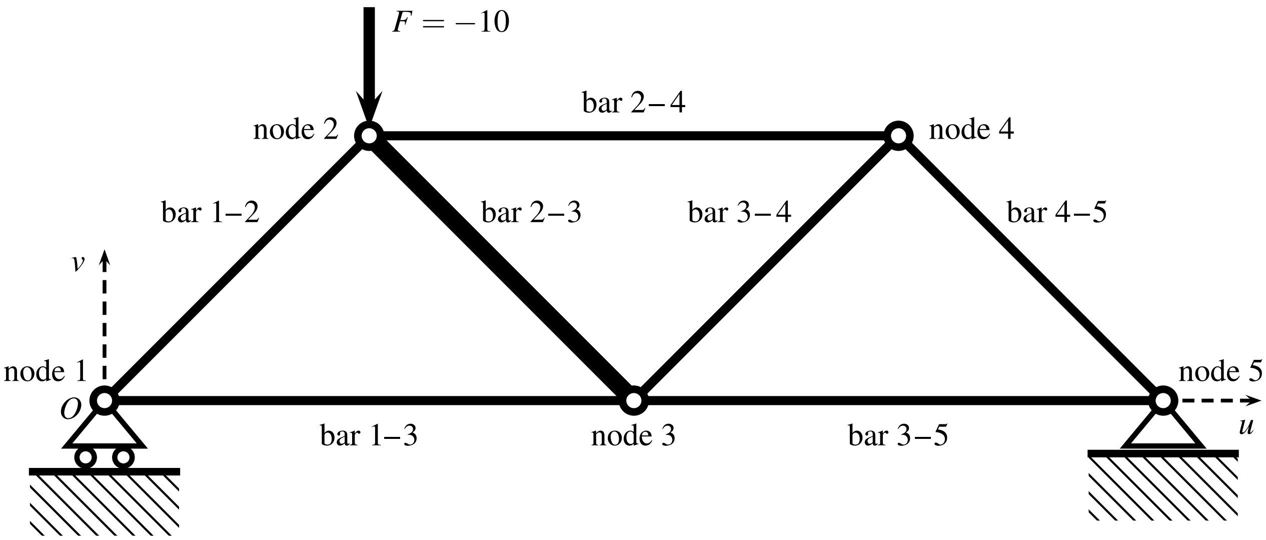

Fig. 3. A seven-member truss loaded by a single vertical force F.

6. A seven-member truss

6.1. The truss problem and the stiffness matrix

As a final application of the present approach based on quantifier elimination here we will consider the problem of a seven-member truss in structural mechanics loaded by a single vertical force F (Fig. 3). This truss problem was originally studied by Kulpa, Pownuk and Skalna [38, Subsection 4.1], next by Skalna [41, Subsection 5.1, Example 1] and [42, Section 4, Example 4.1, pp. 112–114] and, finally, by Elishakoff and Miglis [44, Section 4, pp. 5–7]. The stiffnesses (rigidities) of the bars of the truss are denoted by the symbols s_{ij}, where i and j refer to the nodes at the ends of each bar. Evidently, here $s_{ji} = s_{ij}$ and the related stiffness matrix $\mathbf{K}$ is a symmetric matrix.

Here, following Kulpa, Pownuk and Skalna [38, Subsection 4.1, Fig. 2], Skalna [41, Subsection 5.1, Example 1] and [42, Section 4, Example 4.1, p. 113, Fig. 1] and Elishakoff and Miglis [44, Section 4, p. 5], we assume that the length of each horizontal bar of the truss is $L_{\text{horizontal}} = 2$ (in m) and the length of each skew bar of the truss is $L_{\text{skew}} = \sqrt{2}$ (also in m). Moreover, we similarly assume that the cross-sectional area of each bar is $A_0 = 5$ (in 10^{-3} m^2) and the modulus of elasticity (Young's modulus) of each bar is $E_0 = 200$ (in GPa, ASTM A36 steel). Then $s_{ij} = A_0E_0/L_{ij}$, where L_{ij} is the length of the bar connecting the nodes i and j of the truss. Next, here it is assumed that the stiffness s_{23} of the bar 2–3 of the present truss (Fig. 3, emphasized bar) is an uncertain parameter of the form

$$s_{23} = \frac{A_0E_0}{L_{\text{skew}}}\left(1+\frac{q}{10}\right) \quad \text{with the parameter} \quad q \in [-1, 1] \tag{76}$$

exactly as has been already assumed by Elishakoff and Miglis [44, p. 5, Eq. (33)] (level of uncertainty 10%), but here with the use of the parameter $q \in [-1, 1]$ instead of the parameter t in the trigonometric function $\sin t$ with $t \in [-\pi/2, \pi/2]$ having been preferred by Elishakoff and Miglis [44, p. 5, Eq. (33)]. Now, using the aforementioned values of $L_{\text{horizontal}}$, L_{skew}, A_0 and E_0, we directly find the expressions of the seven stiffnesses s_{ij} of the bars of the present truss (Fig. 3), which are

$$s_{13} = s_{35} = s_{24} = \frac{A_0E_0}{L_{\text{horizontal}}} = \frac{5\cdot 200}{2} = 500, \tag{77}$$

$$s_{12} = s_{34} = s_{45} = \frac{A_0E_0}{L_{\text{skew}}} = \frac{5\cdot 200}{\sqrt{2}} = 500\sqrt{2}, \tag{78}$$

$$s_{23} = \frac{A_0E_0}{L_{\text{skew}}}\left(1+\frac{q}{10}\right) = \frac{5\cdot 200}{\sqrt{2}}\left(1+\frac{q}{10}\right) = 500\sqrt{2}\left(1+\frac{q}{10}\right) \quad \text{with} \quad q \in [-1, 1]. \tag{79}$$

Additionally, the vector of unknowns **X** in the present truss problem (Fig. 3) consists of the seven nodal displacements u_1 (of node 1), u_2 and v_2 (of node 2), u_3 and v_3 (of node 3) and u_4 and v_4 (of node 4), where u_1, u_2, u_3 and u_4 refer to horizontal displacements whereas v_2, v_3 and v_4 to vertical displacements, i.e.

$$\mathbf{X} = \{u_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T. \tag{80}$$

Next, the loading vector (vector of forces) **F** of the present seven-member truss (Fig. 3) is given by

$$\mathbf{F} = \{0 \quad 0 \quad -10 \quad 0 \quad 0 \quad 0 \quad 0\}^T. \tag{81}$$

This load corresponds to the single vertical force $F = -10$ (in kN), which is applied at the node 2 (Fig. 3) of the present truss with a negative direction (downwards) exactly as has been also assumed by Skalna [41, Subsection 5.1, Example 1, Fig. 1] and [42, Section 4, Example 4.1, p. 112] as well as by Elishakoff and Miglis [44, Section 4, p. 5, Fig. 2 and also p. 7, Eq. (37)].

Naturally, the related system of linear algebraic equations has the classical form [41, Section 5, Eq. (20)], [44, p. 3, Eq. (13)]

$$\mathbf{K}\mathbf{X} = \mathbf{F}, \tag{82}$$

but with the stiffness s_{23} of the bar 2–3 of the truss (Fig. 3) having being assumed here to be an uncertain, interval variable because of Eqs. (76) and (79) because the parameter q belongs to the interval $[-1,1]$ ($q \in [-1,1]$) in these equations.

As far as the stiffness matrix **K** of the present truss (Fig. 3) is concerned, this matrix has the form [41, Subsection 5.1, Example 1], [42, Section 4, Example 4.1, pp. 112–113], [44, p. 6, Eq. (34)]

$$\mathbf{K} = \begin{bmatrix} \frac{s_{12}}{2}+s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} & 0 & 0 & 0 \\ -\frac{s_{12}}{2} & \frac{s_{12}+s_{23}}{2}+s_{24} & \frac{s_{12}-s_{23}}{2} & -\frac{s_{23}}{2} & \frac{s_{23}}{2} & -s_{24} & 0 \\ -\frac{s_{12}}{2} & \frac{s_{12}-s_{23}}{2} & \frac{s_{12}+s_{23}}{2} & \frac{s_{23}}{2} & -\frac{s_{23}}{2} & 0 & 0 \\ -s_{13} & -\frac{s_{23}}{2} & \frac{s_{23}}{2} & s_{13}+\frac{s_{34}+s_{23}}{2}+s_{35} & \frac{s_{34}-s_{23}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ 0 & \frac{s_{23}}{2} & -\frac{s_{23}}{2} & \frac{s_{34}-s_{23}}{2} & \frac{s_{34}+s_{23}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ 0 & -s_{24} & 0 & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} & s_{24}+\frac{s_{34}+s_{45}}{2} & 0 \\ 0 & 0 & 0 & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} & 0 & \frac{s_{34}+s_{45}}{2} \end{bmatrix}. \tag{83}$$

With the present values of the stiffnesses (rigidities) s_{ij} in Eqs. (77), (78) and (79) of the seven bars of the present truss (Fig. 3) the above stiffness matrix **K** in Eq. (83) takes the parametric form

$$\mathbf{K} = 250 \times \begin{bmatrix} 2+\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -2 & 0 & 0 & 0 \\ -\sqrt{2} & q^*+2\sqrt{2}+2 & -q^* & -q^*-\sqrt{2} & q^*+\sqrt{2} & -2 & 0 \\ -\sqrt{2} & -q^* & q^*+2\sqrt{2} & q^*+\sqrt{2} & -q^*-\sqrt{2} & 0 & 0 \\ -2 & -q^*-\sqrt{2} & q^*+\sqrt{2} & q^*+2\sqrt{2}+4 & -q^* & -\sqrt{2} & -\sqrt{2} \\ 0 & q^*+\sqrt{2} & -q^*-\sqrt{2} & -q^* & q^*+2\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & 0 & -\sqrt{2} & -\sqrt{2} & 2+2\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & 0 & 2\sqrt{2} \end{bmatrix}, \tag{84}$$

where, for convenience, the new interval parameter q^* is defined (in terms of the original interval parameter q in Eqs. (76) and (79), $q \in [-1,1]$) as

$$q^* := \frac{q}{5\sqrt{2}} = \frac{\sqrt{2}q}{10} = 0.1\sqrt{2}q. \tag{85}$$

Evidently, the matrix $\mathbf{K}$ in Eq. (84) is a symmetric matrix as it should be as a stiffness matrix.

Next, the inverse matrix $\mathbf{K}^{-1}$ of the stiffness matrix $\mathbf{K}$ is easily computed with the help of *Mathematica* [2] and its elements are those computed and displayed in detail by Elishakoff and Miglis [44, p. 6, Eqs. (36)]. For the sake of space the expressions of these elements are not repeated here.

For $q = 0$ (and $q^* = 0$ too because of Eq. (85)) the above stiffness matrix $\mathbf{K}$ in Eq. (84) takes its simpler and now purely numerical form (its central form)

$$\mathbf{K}_c = 250 \times \begin{bmatrix} 2+\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -2 & 0 & 0 & 0 \\ -\sqrt{2} & 2\sqrt{2}+2 & 0 & -\sqrt{2} & \sqrt{2} & -2 & 0 \\ -\sqrt{2} & 0 & 2\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ -2 & -\sqrt{2} & \sqrt{2} & 2\sqrt{2}+4 & 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} & 0 & 2\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & 0 & -\sqrt{2} & -\sqrt{2} & 2+2\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & 0 & 2\sqrt{2} \end{bmatrix}. \tag{86}$$

The inverse matrix $\mathbf{K}_c^{-1}$ of the above central stiffness matrix $\mathbf{K}_c$ (for $q = q^* = 0$) is easily found to have the form

$$\mathbf{K}_c^{-1} = \frac{1}{4000} \times \begin{bmatrix} 16 & 8 & 8 & 8 & 8 & 8 & 8 \\ 8 & 7+2\sqrt{2} & 1 & 6 & 0 & 3+2\sqrt{2} & 3 \\ 8 & 1 & 7+6\sqrt{2} & 2 & 4(2+\sqrt{2}) & 5 & 5+2\sqrt{2} \\ 8 & 6 & 2 & 8 & 4 & 6 & 6 \\ 8 & 0 & 4(2+\sqrt{2}) & 4 & 4(3+2\sqrt{2}) & 8 & 4(2+\sqrt{2}) \\ 8 & 3+2\sqrt{2} & 5 & 6 & 8 & 7+2\sqrt{2} & 7 \\ 8 & 3 & 5+2\sqrt{2} & 6 & 4(2+\sqrt{2}) & 7 & 7+6\sqrt{2} \end{bmatrix}. \tag{87}$$

On the other hand, the difference $\Delta\mathbf{K} := \mathbf{K} - \mathbf{K}_c$ is also easily found to have the simple parametric form

$$\Delta\mathbf{K} = 25\sqrt{2} \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & -q & -q & q & 0 & 0 \\ 0 & -q & q & q & -q & 0 & 0 \\ 0 & -q & q & q & -q & 0 & 0 \\ 0 & q & -q & -q & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{88}$$

This form is clear from Eqs. (84) and (86) for the stiffness matrices $\mathbf{K}$ and $\mathbf{K}_c$ (for $q = q^* = 0$), respectively, as well as from the fact that $q^* := q/(5\sqrt{2})$, Eq. (85).

6.2. The central solution and the first perturbed solution

Now we can easily compute the central solution

$$\mathbf{X}_c = \{u_{1c} \quad u_{2c} \quad v_{2c} \quad u_{3c} \quad v_{3c} \quad u_{4c} \quad v_{4c}\}^T, \tag{89}$$

which corresponds to the central value (the mean value) $q = 0$ of the interval parameter $q \in [-1, 1]$, and the related central displacements (the mean displacements) u_{1c}, u_{2c}, v_{2c}, u_{3c}, v_{3c}, u_{4c} and v_{4c} of the nodes of the present seven-member truss (again for $q = 0$). This solution easily results from the related matrix equation (on the basis of Eqs. (82) for the original matrix equations, (86) for the central stiffness matrix $\mathbf{K}_c$ and (81) for the loading vector $\mathbf{F}$). This matrix equation has the form

$$\mathbf{K}_c\mathbf{X}_c = \mathbf{F} \quad \text{whence} \quad \mathbf{X}_c = \mathbf{K}_c^{-1}\mathbf{F}. \tag{90}$$

The resulting vector of unknowns (here vector of nodal displacements) $\mathbf{X}_c$ has the form

$$\mathbf{X}_c = \frac{1}{400}\{-8 \quad -1 \quad -7-6\sqrt{2} \quad -2 \quad -8-4\sqrt{2} \quad -5 \quad -5-2\sqrt{2}\}^T \tag{91}$$

and numerically the corresponding form

$$\mathbf{X}_c \approx \{-0.02 \ -0.0025 \ -0.03871320 \ -0.005 \ -0.03414214 \ -0.0125 \ -0.01957107\}^T. \tag{92}$$

Therefore, the central values (for $q = 0$) of the nodal displacements are

$$u_{1c} = -\frac{1}{50}, \quad u_{2c} = -\frac{1}{400}, \quad v_{2c} = -\frac{1}{400}(7+6\sqrt{2}), \quad u_{3c} = -\frac{1}{200},$$
$$v_{3c} = -\frac{1}{100}(2+\sqrt{2}), \quad u_{4c} = -\frac{1}{80}, \quad v_{4c} = -\frac{1}{400}(5+2\sqrt{2}). \tag{93}$$

Now we proceed to the method of perturbations with $q \neq 0$. This method is very well known and it is described in detail by Qiu, Chen and Song [35] and Qiu and Elishakoff [37]. (These two papers concern structures with an emphasis on interval analysis exactly as is here the case; see also the related paper by McWilliam [39].) The book by Deif [106, Chapter 6, Sections 6.1 and 6.2, pp. 212–222] is also of interest. For the first perturbation we have the well-known formula

$$\Delta\mathbf{X}_1 = -\mathbf{K}_c^{-1}\Delta\mathbf{K}\mathbf{X}_c. \tag{94}$$

By using Eq. (87) for the inverse central matrix $\mathbf{K}_c^{-1}$, Eq. (88) for the difference $\Delta\mathbf{K}$ and Eq. (91) for the central solution $\mathbf{X}_c$ (for $q = 0$), we directly find that

$$\Delta\mathbf{X}_1 = \frac{1}{4000\sqrt{2}}\{0 \quad -q \quad q \quad 0 \quad -2q \quad -q \quad -q\}^T. \tag{95}$$

Hence, for the first perturbation we have

$$\Delta u_1 = 0, \quad \Delta u_2 = -\frac{q}{4000\sqrt{2}}, \quad \Delta v_2 = \frac{q}{4000\sqrt{2}}, \quad \Delta u_3 = 0,$$
$$\Delta v_3 = -\frac{q}{2000\sqrt{2}}, \quad \Delta u_4 = -\frac{q}{4000\sqrt{2}}, \quad \Delta v_4 = -\frac{q}{4000\sqrt{2}}. \tag{96}$$

The related approximation $\mathbf{X}_1$ to the vector of unknowns $\mathbf{X}$ in Eq. (80) is determined by using the well-known formula

$$\mathbf{X}_1 = \mathbf{X}_c + \Delta\mathbf{X}_1. \tag{97}$$

The resulting expressions of the seven approximate nodal displacements $u_{1,p1}$, $u_{2,p1}$, $v_{2,p1}$, $u_{3,p1}$, $v_{3,p1}$, $u_{4,p1}$ and $v_{4,p1}$, which are the elements of the vector of unknowns $\mathbf{X}_1$ in Eq. (97) computed for the first perturbation (the perturbed solution of the first perturbation), have the following forms:

$$u_{1,p1} = -\frac{1}{50}, \quad u_{2,p1} = -\frac{\sqrt{2}q+20}{8000}, \quad v_{2,p1} = \frac{\sqrt{2}q-20(7+6\sqrt{2})}{8000}, \quad u_{3,p1} = -\frac{1}{200},$$
$$v_{3,p1} = -\frac{\sqrt{2}q+40(2+\sqrt{2})}{4000}, \quad u_{4,p1} = -\frac{\sqrt{2}q+100}{8000}, \quad v_{4,p1} = -\frac{\sqrt{2}q+20(5+2\sqrt{2})}{8000}. \tag{98}$$

From the above results for the first perturbed solution we observe that the displacements $u_{1,p1}$ and $u_{3,p1}$ have crisp (deterministic) values, i.e. values without any uncertainty (being independent of the parameter $q \in [-1,1]$). This fact is in complete agreement with the closed-form solution of the present truss problem obtained by Elishakoff and Miglis [44, Section 4, p. 7, Eqs. (39) and (40)].

6.3. The twentieth perturbed solution

In quite a similar manner, we can proceed to the determination of additional perturbed solutions. The related formulae are well known [35, p. 5, Eq. (32)], [37, p. 364, Eq. (32)] and have the forms

$$\Delta\mathbf{X}_k = -\mathbf{K}_c^{-1}\Delta\mathbf{K}\,\Delta\mathbf{X}_{k-1}, \quad k = 2,3,\ldots, \tag{99}$$

which are written here in a somewhat different notation. The components of the vector $\Delta\mathbf{X}_k$ are

$$\Delta\mathbf{X}_k = \{\Delta u_{1,k} \quad \Delta u_{2,k} \quad \Delta v_{2,k} \quad \Delta u_{3,k} \quad \Delta v_{3,k} \quad \Delta u_{4,k} \quad \Delta v_{4,k}\}^T. \tag{100}$$

Here we will pay attention only to the twentieth perturbed solution, which we will use below.

On the basis of the above formulae (99), by using the related *Mathematica* command

```
Table[ΔX[k]=-KcI.ΔK.ΔX[k-1]//Simplify,{k,2,20}]                    [c44]
```

we easily determine the vectors $\Delta\mathbf{X}_k$ for $k = 2,3,\ldots,20$. For example, for $k = 2$ we find that

$$\Delta\mathbf{X}_2 = \left\{0 \quad \frac{q^2\cdot 10^{-4}}{4\sqrt{2}} \quad -\frac{q^2\cdot 10^{-4}}{4\sqrt{2}} \quad 0 \quad \frac{q^2\cdot 10^{-4}}{2\sqrt{2}} \quad \frac{q^2\cdot 10^{-4}}{4\sqrt{2}} \quad \frac{q^2\cdot 10^{-4}}{4\sqrt{2}}\right\}^T. \tag{101}$$

Similarly, for $k = 20$ we find that

$$\Delta\mathbf{X}_{20} = \left\{0 \quad \frac{q^{20}\cdot 10^{-22}}{4\sqrt{2}} \quad -\frac{q^{20}\cdot 10^{-22}}{4\sqrt{2}} \quad 0 \quad \frac{q^{20}\cdot 10^{-22}}{2\sqrt{2}} \quad \frac{q^{20}\cdot 10^{-22}}{4\sqrt{2}} \quad \frac{q^{20}\cdot 10^{-22}}{4\sqrt{2}}\right\}^T. \tag{102}$$

Now, as far as the nodal displacements $u_1, u_2, v_2, u_3, v_3, u_4$ and v_4 in the vector of unknowns $\mathbf{X}$ are concerned, they can be determined for $k = 20$ very easily on the basis of the vectors $\Delta\mathbf{X}_k$. The related perturbation-based approximations are denoted as $u_{1,p20}, u_{2,p20}, v_{2,p20}, u_{3,p20}, v_{3,p20}, u_{4,p20}$ and $v_{4,p20}$ and they are determined by using the formulae

$$u_{1,p20} = u_{1c} + \sum_{k=1}^{20} \Delta u_{1,k}, \tag{103}$$

$$u_{2,p20} = u_{2c} + \sum_{k=1}^{20} \Delta u_{2,k}, \tag{104}$$

$$v_{2,p20} = v_{2c} + \sum_{k=1}^{20} \Delta v_{2,k}, \tag{105}$$

$$u_{3,p20} = u_{3c} + \sum_{k=1}^{20} \Delta u_{3,k}, \tag{106}$$

$$v_{3,p20} = v_{3c} + \sum_{k=1}^{20} \Delta v_{3,k}, \tag{107}$$

$$u_{4,p20} = u_{4c} + \sum_{k=1}^{20} \Delta u_{4,k}, \tag{108}$$

$$v_{4,p20} = v_{4c} + \sum_{k=1}^{20} \Delta v_{4,k}, \tag{109}$$

where u_{1c}, u_{2c}, v_{2c}, u_{3c}, v_{3c}, u_{4c} and v_{4c} refer to the central (mean) values of these nodal displacements computed for $q = 0$. These central (mean) values are already available from Eqs. (93).

The resulting polynomial approximations $u_{1,p20}, u_{2,p20}, v_{2,p20}, u_{3,p20}, v_{3,p20}, u_{4,p20}$ and $v_{4,p20}$ have the following forms (here with $q \in [-1,1]$) directly found with the help of *Mathematica* [2]:

$$u_{1,p20} = -\frac{1}{50}, \tag{110}$$

$$u_{2,p20} = \frac{1}{4\sqrt{2}} P_{20}(q) - \frac{1}{400}, \tag{111}$$

$$v_{2,p20} = -\frac{1}{4\sqrt{2}} P_{20}(q) - \frac{1}{400}(7+6\sqrt{2}), \tag{112}$$

$$u_{3,p20} = -\frac{1}{200}, \tag{113}$$

$$v_{3,p20} = \frac{1}{2\sqrt{2}} P_{20}(q) - \frac{1}{100}(2+\sqrt{2}), \tag{114}$$

$$u_{4,p20} = \frac{1}{4\sqrt{2}} P_{20}(q) - \frac{1}{80}, \tag{115}$$

$$v_{4,p20} = \frac{1}{4\sqrt{2}} P_{20}(q) - \frac{1}{400}(5+2\sqrt{2}), \tag{116}$$

where the twentieth-degree perturbation-based polynomial $P_{20}(q)$ was found to have the form

$$\begin{aligned} P_{20}(q) = {} & q^{20}\cdot 10^{-22} - q^{19}\cdot 10^{-21} + q^{18}\cdot 10^{-20} - q^{17}\cdot 10^{-19} + q^{16}\cdot 10^{-18} - q^{15}\cdot 10^{-17} \\ & + q^{14}\cdot 10^{-16} - q^{13}\cdot 10^{-15} + q^{12}\cdot 10^{-14} - q^{11}\cdot 10^{-13} + q^{10}\cdot 10^{-12} - q^{9}\cdot 10^{-11} \\ & + q^{8}\cdot 10^{-10} - q^{7}\cdot 10^{-9} + q^{6}\cdot 10^{-8} - q^{5}\cdot 10^{-7} + q^{4}\cdot 10^{-6} - q^{3}\cdot 10^{-5} \\ & + q^{2}\cdot 10^{-4} - q\cdot 10^{-3}. \end{aligned} \tag{117}$$

6.4. Comparison with the Taylor–Maclaurin series of the closed-form solution

On the other hand, the parametric interval system of linear algebraic equations (82), i.e. the system $\mathbf{KX} = \mathbf{F}$, with vector of unknowns $\mathbf{X} = \{u_1\ \ u_2\ \ v_2\ \ u_3\ \ v_3\ \ u_4\ \ v_4\}^T$ in Eq. (80), loading vector $\mathbf{F} = \{0\ \ 0\ \ -10\ \ 0\ \ 0\ \ 0\ \ 0\}^T$ in Eq. (81) and coefficient matrix the parametric stiffness matrix $\mathbf{K}$ in Eq. (84) is a simple interval system of linear algebraic equations (although, evidently, this is not generally the case in structural mechanics) and, therefore, its closed-form solution can easily be found using the `Solve` command of *Mathematica*. The resulting closed-form solution has the form

$$u_1 = -\frac{1}{50} = -0.02, \tag{118}$$

$$u_2 = -\frac{(2+\sqrt{2})q+20}{800(q+10)} = -\frac{0.00426777q+0.025}{q+10}, \tag{119}$$

$$v_2 = -\frac{1}{800}\left(\frac{10\sqrt{2}}{q+10} + 11\sqrt{2} + 14\right) = -\frac{0.0369454q+0.387132}{q+10}, \tag{120}$$

$$u_3 = -\frac{1}{200} = -0.005, \tag{121}$$

$$v_3 = -\frac{(8+5\sqrt{2})q+40(2+\sqrt{2})}{400(q+10)} = -\frac{0.0376777q+0.341421}{q+10}, \tag{122}$$

$$u_4 = -\frac{(10+\sqrt{2})q+100}{800(q+10)} = -\frac{0.0142678q+0.125}{q+10}, \tag{123}$$

$$v_4 = -\frac{(2+\sqrt{2})q+8\sqrt{2}+20}{160(q+10)} = -\frac{0.0213388q+0.195711}{q+10}. \tag{124}$$

This solution is in agreement with that already found by Elishakoff and Miglis [44, p. 7, Eq. (39)], who used their approach based on the use of the trigonometric function $\sin t$ (with $t \in [-\pi/2, \pi/2]$).

From the above closed-form solution of the parametric system $\mathbf{KX} = \mathbf{F}$ in Eq. (82) at first we observe that $u_1 = u_{1,p20}$ in Eq. (110) and also that $u_3 = u_{3,p20}$ in Eq. (113) as is really expected for these crisp (deterministic) quantities. Next, using the `Series` command of *Mathematica* we can directly verify that the five approximations $u_{2,p20}$, $v_{2,p20}$, $v_{3,p20}$, $u_{4,p20}$ and $v_{4,p20}$ in Eqs. (111), (112), (114), (115) and (116), respectively, which have been obtained here using the perturbation method, coincide with the Taylor–Maclaurin series $u_{2,s}$, $v_{2,s}$, $v_{3,s}$, $u_{4,s}$ and $v_{4,s}$ of the closed-form solutions u_2, v_2, v_3, u_4 and v_4 in the above Eqs. (119), (120), (122), (123) and (124), respectively, i.e.

$$u_{2,p20} = u_{2,s}, \quad v_{2,p20} = v_{2,s}, \quad v_{3,p20} = v_{3,s}, \quad u_{4,p20} = u_{4,s} \quad \text{and} \quad v_{4,p20} = v_{4,s}. \tag{125}$$

Naturally, here we preferred to use the perturbation method instead of the alternative method based on Taylor–Maclaurin series simply because the perturbation method is of general validity and it does not require the closed-form solution of the parametric interval system of linear algebraic equations $\mathbf{KX} = \mathbf{F}$ (here with parameter $q \in [-1, 1]$). This closed-form solution is often unavailable because its computation may be a difficult or even impossible task (even with the use of *Mathematica*) for many unknowns $\mathbf{X}$ in this parametric interval system $\mathbf{KX} = \mathbf{F}$ and/or for many parameters q_i.

For this reason from now on we will employ the approximations $u_{2,p20}$, $v_{2,p20}$, $v_{3,p20}$, $u_{4,p20}$ and $v_{4,p20}$ instead of the corresponding closed-form solutions u_2, v_2, v_3, u_4 and v_4, respectively, thus putting an emphasis on the use of the perturbation method, which is of general validity in comparison with closed-form solutions. This approach is somewhat analogous to the approach already adopted in the previous three sections, where we also used polynomial approximations (but not the perturbation method) to the transcendental functions appearing in the functions studied there.

In fact, for the five approximate parametric solutions $u_{2,p20}$, $v_{2,p20}$, $v_{3,p20}$, $u_{4,p20}$ and $v_{4,p20}$ in Eqs. (111), (112), (114), (115) and (116), respectively, and for the interval $[-1, 1]$ employed here for the parameter q (having assumed a 10% level of uncertainty in the stiffness s_{23} of the bar 2–3 of the seven-member truss of Fig. 3 with $q \in [-1, 1]$) we can use the existentially quantified formula

$$\exists q \in [-1, 1] \text{ such that } u_2 = u_{2,p20} + r. \tag{126}$$

The related quantifier elimination command in *Mathematica* has the simple form

```
Reduce[Exists[q, -1 ≤ q ≤ 1, u2 == u2p20 + r], Reals]//N          [c45]
```

Then for the nodal displacement u_2 we found the QFF (i.e., essentially, the related interval)

$$-1.60706 \cdot 10^{-24} \le r \le 1.96419 \cdot 10^{-24}, \text{ i.e. } R_{u_2} = [-1.60706 \cdot 10^{-24}, 1.96419 \cdot 10^{-24}]. \tag{127}$$

Completely analogously, we found the QFFs (i.e., essentially, the related intervals) for the errors concerning the four approximate parametric nodal displacements $v_{2,p20}$, $v_{3,p20}$, $u_{4,p20}$ and $v_{4,p20}$ in Eqs. (112), (114), (115) and (116), respectively. These intervals are

$$-1.96419 \cdot 10^{-24} \le r \le 1.60706 \cdot 10^{-24}, \text{ i.e. } R_{v_2} = [-1.96419 \cdot 10^{-24}, 1.60706 \cdot 10^{-24}], \tag{128}$$

$$-3.21412 \cdot 10^{-24} \le r \le 3.92837 \cdot 10^{-24}, \text{ i.e. } R_{v_3} = [-3.21412 \cdot 10^{-24}, 3.92837 \cdot 10^{-24}] \tag{129}$$

for the nodal displacements v_2 and v_3 whereas we also found that $R_{u_4} = R_{v_4} = R_{u_2}$ for the nodal displacements u_4 and v_4. The conclusion is simply that the assumed polynomial approximations

$$u_{2,p20} \approx u_2, \quad v_{2,p20} \approx v_2, \quad v_{3,p20} \approx v_3, \quad u_{4,p20} \approx u_4, \quad v_{4,p20} \approx v_4 \tag{130}$$

are really excellent approximations. Therefore, these approximations will be used below during the application of the method of quantifier elimination to the present truss problem for the derivation of simple interval-based polynomial approximations to the nodal displacements u_2, v_2, v_3, u_4 and v_4.

6.5. The simple perturbation-based polynomial approximations to be used here

Here we will make use of simple perturbation-based polynomial approximations to the nodal displacements u_1, u_2, v_2, u_3, v_3, u_4 and v_4 with terms only up to q^2. These approximations are obtained by using formulae completely analogous to Eqs. (103)–(109), but now with k up to $k=2$ in the related sums (instead of k up to $k=20$ in Subsection 6.3). In this way, we find that

$$u_{1,p2} = u_{1c} + \sum_{k=1}^{2} \Delta u_{1,k} = -\frac{1}{50}, \tag{131}$$

$$u_{2,p2} = u_{2c} + \sum_{k=1}^{2} \Delta u_{2,k} = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400}, \tag{132}$$

$$v_{2,p2} = v_{2c} + \sum_{k=1}^{2} \Delta v_{2,k} = -\frac{q^2}{40000\sqrt{2}} + \frac{q}{4000\sqrt{2}} - \frac{1}{400}(7+6\sqrt{2}), \tag{133}$$

$$u_{3,p2} = u_{3c} + \sum_{k=1}^{2} \Delta u_{3,k} = -\frac{1}{200}, \tag{134}$$

$$v_{3,p2} = v_{3c} + \sum_{k=1}^{2} \Delta v_{3,k} = \frac{q^2}{20000\sqrt{2}} - \frac{q}{2000\sqrt{2}} - \frac{1}{100}(2+\sqrt{2}), \tag{135}$$

$$u_{4,p2} = u_{4c} + \sum_{k=1}^{2} \Delta u_{4,k} = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{80}, \tag{136}$$

$$v_{4,p2} = v_{4c} + \sum_{k=1}^{2} \Delta v_{4,k} = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400}(5+2\sqrt{2}). \tag{137}$$

Obviously, the nodal displacements $u_{1,p2}$ and $u_{3,p2}$ have crisp (deterministic) values (exact values). Therefore, we will ignore these two nodal displacements during quantifier eliminations below.

6.6. The interval-based quadratic polynomial approximations and the related intervals

Now, based on the above approximations to the nodal displacements $u_{2,p2}$, $v_{2,p2}$, $v_{3,p2}$, $u_{4,p2}$ and $v_{4,p2}$ in Eqs. (132), (133), (135), (136) and (137), respectively, we will determine the intervals R of the parameters r (with $r \in R$) which will permit us to have approximate but very reliable interval-based quadratic polynomial approximations (interval enclosures) to these displacements.

At first, we consider the five interval-based quadratic polynomial approximations

$$u_{2,p2}^{(r,1)} = u_{2,p2} + r = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400} + r, \tag{138}$$

$$v_{2,p2}^{(r,1)} = v_{2,p2} + r = -\frac{q^2}{40000\sqrt{2}} + \frac{q}{4000\sqrt{2}} - \frac{1}{400}(7+6\sqrt{2}) + r, \tag{139}$$

$$v_{3,p2}^{(r,1)} = v_{3,p2} + r = \frac{q^2}{20000\sqrt{2}} - \frac{q}{2000\sqrt{2}} - \frac{1}{100}(2+\sqrt{2}) + r, \tag{140}$$

$$u_{4,p2}^{(r,1)} = u_{4,p2} + r = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{80} + r, \tag{141}$$

$$v_{4,p2}^{(r,1)} = v_{4,p2} + r = \frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400}(5+2\sqrt{2}) + r. \tag{142}$$

For the determination of the intervals (ranges) R of the interval parameters r in these approximations (with $q \in [-1,1]$ as was already mentioned) we can use the existentially quantified formula

$$\exists q \in [-1,1] \text{ such that } u_{2,p20} = u_{2,p2}^{(r,1)} + r \tag{143}$$

for the interval-related nodal displacement u_2 and analogously for the four interval-related nodal displacements v_2, v_3, u_4 and v_4. The related quantifier elimination command in *Mathematica* is

```
Reduce[Exists[q, -1 ≤ q ≤ 1, u2p20 == u2p2+r], Reals]//N                    [c46]
```

for u_2 and analogously for the nodal displacements v_2, v_3, u_4 and v_4. The resulting QFFs (quantifier-free formulae) for the parameters r and the related intervals R (with $r \in R$) have the forms

$$-1.60706 \cdot 10^{-6} \leq r \leq 1.96419 \cdot 10^{-6}, \quad \text{i.e.} \quad R = [-1.60706 \cdot 10^{-6}, 1.96419 \cdot 10^{-6}] \tag{144}$$

for the three nodal displacements u_2, u_4 and v_4,

$$-1.96419 \cdot 10^{-6} \leq r \leq 1.60706 \cdot 10^{-6}, \quad \text{i.e.} \quad R = [-1.96419 \cdot 10^{-6}, 1.60706 \cdot 10^{-6}] \tag{145}$$

for the nodal displacement v_2 and

$$-3.21412 \cdot 10^{-6} \leq r \leq 3.92837 \cdot 10^{-6}, \quad \text{i.e.} \quad R = [-3.21412 \cdot 10^{-6}, 3.92837 \cdot 10^{-6}] \tag{146}$$

for the nodal displacement v_3.

In a similar way, we consider the five generalized interval-based quadratic polynomial approximations

$$u_{2,p2}^{(r,2)} = (1+r)u_{2,p2} = (1+r)\Big(\frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400}\Big), \tag{147}$$

$$v_{2,p2}^{(r,2)} = (1+r)v_{2,p2} = (1+r)\Big(-\frac{q^2}{40000\sqrt{2}} + \frac{q}{4000\sqrt{2}} - \frac{1}{400}(7+6\sqrt{2})\Big), \tag{148}$$

$$v_{3,p2}^{(r,2)} = (1+r)v_{3,p2} = (1+r)\Big(\frac{q^2}{20000\sqrt{2}} - \frac{q}{2000\sqrt{2}} - \frac{1}{100}(2+\sqrt{2})\Big), \tag{149}$$

$$u_{4,p2}^{(r,2)} = (1+r)u_{4,p2} = (1+r)\Big(\frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{80}\Big), \tag{150}$$

$$v_{4,p2}^{(r,2)} = (1+r)v_{4,p2} = (1+r)\Big(\frac{q^2}{40000\sqrt{2}} - \frac{q}{4000\sqrt{2}} - \frac{1}{400}(5+2\sqrt{2})\Big). \tag{151}$$

For the determination of the intervals (ranges) R of the interval parameters r in this second case, we work exactly as previously using again the method of quantifier elimination. The resulting QFFs (quantifier-free formulae) for the five parameters r and the related intervals R (with $r \in R$) are now

$$-8.51940 \cdot 10^{-4} \leq r \leq 6.04363 \cdot 10^{-4}, \quad \text{i.e.} \quad R = [-8.51940 \cdot 10^{-4}, 6.04363 \cdot 10^{-4}] \tag{152}$$

for the nodal displacement u_2,

$$-0.416833 \cdot 10^{-4} \leq r \leq 0.504833 \cdot 10^{-4}, \quad \text{i.e.} \quad R = [-0.416833 \cdot 10^{-4}, 0.504833 \cdot 10^{-4}] \tag{153}$$

for the nodal displacement v_2,

$$-1.16385 \cdot 10^{-4} \leq r \leq 0.932702 \cdot 10^{-4}, \quad \text{i.e.} \quad R = [-1.16385 \cdot 10^{-4}, 0.932702 \cdot 10^{-4}] \tag{154}$$

for the nodal displacement v_3,

$$-1.59618 \cdot 10^{-4} \leq r \leq 1.26949 \cdot 10^{-4}, \quad \text{i.e.} \quad R = [-1.59618 \cdot 10^{-4}, 1.26949 \cdot 10^{-4}] \tag{155}$$

for the nodal displacement u_4 and

$$-1.01369 \cdot 10^{-4} \leq r \leq 0.814520 \cdot 10^{-4}, \quad \text{i.e.} \quad R = [-1.01369 \cdot 10^{-4}, 0.814520 \cdot 10^{-4}] \tag{156}$$

for the nodal displacement v_4. Here we remind that the nodal displacements u_1 and u_3 have crisp (deterministic) values unrelated to intervals and to the perturbation method having been used here.

7. Conclusions–discussion

From the above results it is concluded that the method of quantifier elimination permits us to find generalized interval-based polynomial approximations to functions (here interval enclosures for functions) in practical problems of applied mechanics such as (i) the problem of a beam on a Winkler elastic foundation (studied in Section 4), (ii) the problem of free vibrations of a harmonic oscillator with critical damping (studied in Section 5) and (iii) the problem of a seven-member truss with the stiffness of one bar of this truss being an uncertain, an interval variable (studied in Section 6). These problems were successfully studied here with the help of the implementation of quantifier-elimination algorithms (mainly of CAD and virtual substitution) in the computer algebra system *Mathematica* [2] by Strzeboński. This implementation seems to be the best available today, both powerful and user-friendly with elementary commands, for performing quantifier elimination.

By using this approach, quantifier elimination, we have been able to find simple (based on polynomials of a low degree) but simultaneously accurate interval enclosures for the functions under consideration more explicitly (i) the deflection of a beam (Section 4), (ii) the displacement of a harmonic oscillator (Section 5) and (iii) the nodal displacements in a seven-member truss (Section 6).

As has been already mentioned, the present results are based on previous fundamental research results already referenced in Section 1 mainly by Kolev; see, e.g., Ref. [91] for the automatic computation of interval enclosures, but by using a completely different method. Moreover, the present results generalize previous recent results (by using quantifier elimination) concerning the determination of ranges of functions in problems of applied mechanics [18], which also provide interval enclosures for the functions under consideration. But it is clearly understood that the present approach on the basis of interval-based polynomial approximations is much more accurate as far as the related interval enclosures are concerned than the method in Ref. [18] restricted to the determination of ranges of functions. This is a simpler but not accurate special case of the present method.

As is very well known and was already mentioned in Section 1, the method of quantifier elimination for real variables used here has a doubly-exponential computational complexity. This characteristic was proved by Davenport and Heintz in 1988 [8]. Therefore, unfortunately, the use of the present method is restricted to problems with only few variables (both free and quantified) in the quantified formulae (e.g. three to five variables) and it cannot be employed with a large total number of variables. This has been the case in the applications having been studied in the previous sections.

Here we have been interested in the determination of simple interval-based polynomial approximations (interval enclosures) to functions in applied mechanics, i.e. based on initial polynomial approximations of a low degree, but, nevertheless, much more accurate than the simple determination of the ranges of the same functions [18]. Beyond the aforementioned doubly-exponential computational complexity of quantifier elimination another restriction of this approach is that it is generally applicable only to polynomial and rational functions and not to transcendental functions with the exception of few special cases or the use of new variables, e.g. the use of $t = \tan(\theta/2)$ instead of θ.

The approach used here for the avoidance of the latter restriction was simply to employ high-degree polynomial approximations to the transcendental functions with extremely small, negligible errors. In fact, in the previous four sections, we have used such polynomial approximations instead of the transcendental functions under consideration: (i) a Taylor series approximation to the exponential function in Section 3, (ii) Taylor–Maclaurin series approximations instead of transcendental functions in Section 4 and in Section 5 and (iii) perturbation-based approximations in Section 6. Analogous approximations were also used for the basic polynomials in the present interval-based polynomial approximations with the exception of Section 5, where minimax approximations were used. An interesting conclusion of the present results is that the selection of the interval-based polynomial approximation (including the interval variable $r \in R$) on the basis of the same basic polynomial approximation is not of particular importance to the achieved accuracy of the approximation.

References

[1] Pavlović, M. N., Symbolic computation in structural engineering. *Computers and Structures*, **81** (22–23), 2121–2136 (2003). https://doi.org/10.1016/S0045-7949(03)00286-4

[2] Wolfram Research Inc., *Mathematica*, version 7.0.1.0. Wolfram Research Inc., Champaign, Illinois, 2009. https://www.wolfram.com

[3] Ratschan, S., Applications of quantified constraint solving over the reals – bibliography. arXiv: 1205.5571v1. Cornell University Library, Ithaca, New York, 2012. https://arxiv.org/abs/1205.5571v1

[4] Collins, G. E., Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In [5], pp. 85–121. https://doi.org/10.1007/978-3-7091-9459-1_4. Reproduced (with corrections) from the original publication (under the same title, Donnerstagvormittag Hauptvortrag) in: Brakhage, H. (editor), *Automata Theory and Formal Languages*, Proceedings of the 2nd GI Conference (Gesellschaft für Informatik), Kaiserslautern, Germany, May 20–23, 1975. Book series: *Lecture Notes in Computer Science (LNCS)*, Vol. 33, pp. 134–183. Springer, Berlin, 1975. https://doi.org/10.1007/3-540-07407-4_17

[5] Caviness, B. F. and Johnson, J. R. (editors), *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Springer, Wien, New York, 1998. https://doi.org/10.1007/978-3-7091-9459-1

[6] Strzeboński, A., Cylindrical algebraic decomposition using local projections. *Journal of Symbolic Computation*, **76**, 36–64 (2016). https://doi.org/10.1016/j.jsc.2015.11.018

[7] England, M., Bradford, R. and Davenport, J. H., Cylindrical algebraic decomposition with equational constraints. *Journal of Symbolic Computation*, **100**, 38–71 (2020). https://doi.org/10.1016/j.jsc.2019.07.019; also in arXiv (in preprint form): arXiv: 1903.08999. Cornell University Library, Ithaca, New York, 2019. https://arxiv.org/abs/1903.08999

[8] Davenport, J. H. and Heintz, J., Real quantifier elimination is doubly exponential. *Journal of Symbolic Computation*, **5** (1–2), 29–35 (1988). https://doi.org/10.1016/S0747-7171(88)80004-X

[9] Weispfenning, V., The complexity of linear problems in fields. *Journal of Symbolic Computation*, **5** (1–2), 3–27 (1988). https://doi.org/10.1016/S0747-7171(88)80003-8

[10] Loos, R. and Weispfenning, V., Applying linear quantifier elimination. *The Computer Journal*, **36** (5), 450–462 (1993). https://doi.org/10.1093/comjnl/36.5.450

[11] Weispfenning, V., Quantifier elimination for real algebra—the quadratic case and beyond. *Applicable Algebra in Engineering, Communication and Computing*, **8** (2), 85–101 (1997). https://doi.org/10.1007/s002000050055

[12] Wolfram Research Inc., Wolfram Monograph: *Advanced Topics in Algebra: Real Polynomial Systems*. Wolfram Research Inc., Champaign, Illinois, 2014. https://reference.wolfram.com/language/tutorial/RealPolynomialSystems.html

[13] Trott, M., *The Mathematica GuideBook for Symbolics*. Springer, New York, 2006. https://doi.org/10.1007/0-387-28815-5

[14] Ioakimidis, N. I., Application of quantifier elimination to a simple elastic beam finite element below a straight rigid obstacle. *Mechanics Research Communications*, **22** (3), 271–278 (1995). https://doi.org/10.1016/0093-6413(95)00023-K

[15] Ioakimidis, N. I., REDLOG-aided derivation of feasibility conditions in applied mechanics and engineering problems under simple inequality constraints. *Journal of Mechanical Engineering (Strojnícky Časopis)*, **50** (1), 58–69 (1999). Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11096/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11096/1/TR-1998-O3.pdf

[16] Ioakimidis, N. I., Derivation of conditions of complete contact for a beam on a tensionless Winkler elastic foundation with *Mathematica*. *Mechanics Research Communications*, **72**, 64–73 (2016). https://doi.org/10.1016/j.mechrescom.2016.01.007

[17] Ioakimidis, N. I., Derivation of feasibility conditions in classical vibration problems by using the method of quantifier elimination. Technical Report No. TR-2018-Q2, School of Engineering, University of Patras, Patras, Greece, 2018, 16 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11505/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11505/1/TR-2018-Q2.pdf; see also https://doi.org/10.13140/RG.2.2.31616.07688

[18] Ioakimidis, N. I., Computation of ranges of functions in problems of applied mechanics with the computational method of quantifier elimination. Technical Report No. TR-2018-Q3, School of Engineering, University of Patras, Patras, Greece, 2018, 24 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11605/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11605/1/TR-2018-Q3.pdf; see also https://doi.org/10.13140/RG.2.2.13161.13929

[19] Ioakimidis, N. I., Stress concentration factors: determination of ranges of values by the method of quantifier elimination. Technical Report No. TR-2018-Q4, School of Engineering, University of Patras, Patras, Greece, 2018, 22 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/11736/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/11736/1/TR-2018-Q4.pdf; see also https://doi.org/10.13140/RG.2.2.19872.02566

[20] Ioakimidis, N. I., Interval computations in the formulae for the stress intensity factors at crack tips using the method of quantifier elimination. Technical Report No. TR-2019-Q5, School of Engineering, University of Patras, Patras, Greece, 2019, 20 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12153/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12153/1/TR-2019-Q5.pdf; see also https://doi.org/10.13140/RG.2.2.33293.79848

[21] Ioakimidis, N. I., Sharp enclosures of the real roots of the classical parametric quadratic equation with one interval coefficient by the method of quantifier elimination. Technical Report No. TR-2019-Q6, School of Engineering, University of Patras, Patras, Greece, 2019, 30 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12159/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12159/1/TR-2019-Q6.pdf; see also https://doi.org/10.13140/RG.2.2.16516.58248

[22] Ioakimidis, N. I., Sharp bounds based on quantifier elimination in truss and other applied mechanics problems with uncertain, interval forces/loads and other parameters. Technical Report No. TR-2019-Q7, School of Engineering, University of Patras, Patras, Greece, 2019, 45 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12497/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12497/1/TR-2019-Q7.pdf; see also https://doi.org/10.13140/RG.2.2.22662.52803

[23] Ioakimidis, N. I., Symbolic intervals for the unknown quantities in simple applied mechanics problems with the computational method of quantifier elimination. Technical Report No. TR-2019-Q8, School of Engineering, University of Patras, Patras, Greece, 2019, 26 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/12745/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/12745/1/TR-2019-Q8.pdf; see also https://doi.org/10.13140/RG.2.2.26311.24485

[24] Ioakimidis, N. I., Interval computations in various direct and inverse applied mechanics problems related to quantifiers by using the method of quantifier elimination. Technical Report No. TR-2019-Q9, School of Engineering, University of Patras, Patras, Greece, 2019, 34 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/13204/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/13204/1/TR-2019-Q9.pdf; see also https://doi.org/10.13140/RG.2.2.13976.14087

[25] Ioakimidis, N. I., Intervals for the resultants of interval forces with existentially and/or universally quantified formulae with the help of the method of quantifier elimination. Technical Report No. TR-2019-Q10, School of Engineering, University of Patras, Patras, Greece, 2019, 34 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/13205/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/13205/1/TR-2019-Q10.pdf; see also https://doi.org/10.13140/RG.2.2.11338.93125

[26] Ioakimidis, N. I., Determination of intervals in systems of parametric interval linear equilibrium equations in applied mechanics with the method of quantifier elimination. Technical Report No. TR-2020-

Q11, School of Engineering, University of Patras, Patras, Greece, 2020, 30 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/13297/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/13297/1/TR-2020-Q11.pdf; see also https://doi.org/10.13140/RG.2.2.14305.25448

[27] Ioakimidis, N. I., Quantifier-elimination-based interval computations in beam problems studied by using the approximate methods of finite differences and of finite elements. Technical Report No. TR-2020-Q12, School of Engineering, University of Patras, Patras, Greece, 2020, 28 pages. Web page at *Nemertes*, the institutional repository of the University of Patras: https://nemertes.lis.upatras.gr/jspui/handle/10889/13518/?locale=en; web page of the related PDF file at *Nemertes*: https://nemertes.lis.upatras.gr/jspui/bitstream/10889/13518/1/TR-2020-Q12.pdf; see also https://doi.org/10.13140/RG.2.2.28966.96325

[28] Charalampakis, A. E. and Chatzigiannelis, I., Analytical solutions for the minimum weight design of trusses by cylindrical algebraic decomposition. *Archive of Applied Mechanics*, **88** (1–2), 39–49 (2018). https://doi.org/10.1007/s00419-017-1271-8

[29] Moore, R. E., Automatic error analysis in digital computation. Technical Report LMSD-48421, Lockheed Aircraft Corporation, Missiles and Space Division, Sunnyvale, California, 1959, v+59 pages. https://interval.louisiana.edu/Moores_early_papers/Moore_Lockheed.pdf

[30] Moore, R. E., *Interval Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

[31] Moore, R. E., Kearfott, R. B. and Cloud, M. J., *Introduction to Interval Analysis*. SIAM (Society for Industrial and Applied Mathematics), Philadelphia, Pennsylvania, 2009. https://doi.org/10.1137/1.9780898717716

[32] Sunaga, T., Theory of an interval algebra and its application to numerical analysis. *RAAG Memoirs* (Tokyo Research Association of Applied Geometry), **2**, 29–46 (547–564) (1958). http://www-sbras.nsc.ru/interval/Introduction/sunaga.pdf

[33] Beebe, N. H. F., Kearfott, R. B. and Kreinovich, V., A bibliography of publications in *interval computations* and *reliable computing*, version 0.03, 2017. https://www.netlib.org/tex/bib/reliablecomputing.pdf

[34] Dimarogonas, A. D., Interval analysis of vibrating systems. *Journal of Sound and Vibration*, **183** (4), 739–749 (1995). https://doi.org/10.1006/jsvi.1995.0283

[35] Qiu, Z., Chen, S. and Song, D., The displacement bound estimation for structures with an interval description of uncertain parameters. *Communications in Numerical Methods in Engineering*, **12** (1), 1–11 (1996). https://doi.org/10.1002/(SICI)1099-0887(199601)12:1<1::AID-CNM884>3.0.CO;2-N

[36] Qiu, Z., Chen, S. and Elishakoff, I., Bounds of eigenvalues for structures with an interval description of uncertain-but-non-random parameters. *Chaos, Solitons & Fractals*, **7** (3), 425–434 (1996). https://doi.org/10.1016/0960-0779(95)00065-8

[37] Qiu, Z. and Elishakoff, I., Antioptimization of structures with large uncertain-but-non-random parameters via interval analysis. *Computer Methods in Applied Mechanics and Engineering*, **152** (3–4), 361–372 (1998). https://doi.org/10.1016/S0045-7825(96)01211-X

[38] Kulpa, Z., Pownuk, A. and Skalna, I., Analysis of linear mechanical structures with uncertainties by means of interval methods. *Computer Assisted Mechanics and Engineering Sciences*, **5** (4), 443–477 (1998). https://www.infona.pl/resource/bwmeta1.element.baztech-article-BPB2-0001-0113; web page at Semantic Scholar with a link to the related PDF file: https://pdfs.semanticscholar.org/f82c/6cb1de4f0c24bb63b1203ad85ca28993f 68e.pdf; web page at ResearchGate also with a link to the related PDF file: https://www.researchgate.net/publication/2489149

[39] McWilliam, S., Anti-optimisation of uncertain structures using interval analysis. *Computers and Structures*, **79** (4), 421–430 (2001). https://doi.org/10.1016/S0045-7949(00)00143-7

[40] Dessombz, O., Thouverez, F., Laîné, J.-P. and Jézéquel, L., Analysis of mechanical systems using interval computations applied to finite element methods. *Journal of Sound and Vibration*, **239** (5), 949–968 (2001). https://doi.org/10.1006/jsvi.2000.3191

[41] Skalna, I., Methods for solving systems of linear equations of structure mechanics with interval parameters. *Computer Assisted Mechanics and Engineering Sciences*, **10** (3), 281–293 (2003). https://

www.infona.pl/resource/bwmeta1.element.baztech-article-BPB1-0009-0078; web page at ResearchGate with a link to the related PDF file: https://www.researchgate.net/publication/248687657

[42] Skalna, I., A method for outer interval solution of systems of linear equations depending linearly on interval parameters. *Reliable Computing*, **12** (2), 107–120 (2006). https://doi.org/10.1007/s11155-006-4878-y

[43] Elishakoff, I. and Ohsaki, M., *Optimization and Anti-Optimization of Structures Under Uncertainty*. Imperial College Press, London, 2010. https://doi.org/10.1142/9781848164789.

[44] Elishakoff, I. and Miglis, Y., Novel parameterized intervals may lead to sharp bounds. *Mechanics Research Communications*, **44**, 1–8 (2012). https://doi.org/10.1016/j.mechrescom.2012.04.004

[45] Elishakoff, I. and Miglis, Y., Overestimation-free computational version of interval analysis. *International Journal for Computational Methods in Engineering Science and Mechanics*, **13** (5), 319–328 (2012). https://doi.org/10.1080/15502287.2012.683134

[46] Wang, C. and Qiu, Z.-p., Equivalent method for accurate solution to linear interval equations. *Applied Mathematics and Mechanics (English Edition)*, **34** (8), 1031–1042 (2013). https://doi.org/10.1007/s10483-013-1725-6

[47] Gabriele, S. and Varano, V., Influence of the parameterization in the interval solution of elastic beams. *Journal of Structures*, Vol. 2014, Article ID 395213, 5 pages. https://doi.org/10.1155/2014/395213

[48] Behera, D., *Numerical Solution of Static and Dynamic Problems of Imprecisely Defined Structural Systems*. Ph.D. Thesis, Department of Mathematics, National Institute of Technology Rourkela, Rourkela, Odisha, India, 2014, xix+259 pages. http://ethesis.nitrkl.ac.in/6583; web page of the related PDF file: http://ethesis.nitrkl.ac.in/6583/1/Ph._D._Thesis_%40Diptiranjan_11-12-2014.pdf

[49] Santoro, R., Muscolino, G. and Elishakoff, I., Optimization and anti-optimization solution of combined parameterized and improved interval analyses for structures with uncertainties. *Computers and Structures*, **149**, 31–42 (2015). https://doi.org/10.1016/j.compstruc.2014.11.006

[50] Sofi, A., Muscolino, G. and Elishakoff, I., Static response bounds of Timoshenko beams with spatially varying interval uncertainties. *Acta Mechanica*, **226** (11), 3737–3748 (2015). https://doi.org/10.1007/s00707-015-1400-9

[51] Qiu, Z.-P. and Wang, L., The need for introduction of non-probabilistic interval conceptions into structural analysis and design. *Science China – Physics, Mechanics & Astronomy*, **59** (11), Paper No. 114632 (2016). https://doi.org/10.1007/s11433-016-0329-3

[52] Zieniuk, E., Kapturczak, M. and Kużelewski, A., Solving interval systems of equations obtained during the numerical solution of boundary value problems. *Computational and Applied Mathematics*, **35** (2), 629–638 (2016). https://doi.org/10.1007/s40314-014-0209-9

[53] Elishakoff, I., Gabriele, S. and Wang, Y., Generalized Galileo Galilei problem in interval setting for functionally related loads. *Archive of Applied Mechanics*, **86** (7), 1203–1217 (2016). https://doi.org/10.1007/s00419-015-1086-4

[54] Popova, E. D., Interval model of equilibrium equations in mechanics. In: Freitag, S., Muhanna, R. L. and Mullen, R. L. (editors), *Proceedings of the 7th International Workshop on Reliable Engineering Computing (REC), Computing with Polymorphic Uncertain Data*, REC 2016, Ruhr University Bochum (Ruhr Universität Bochum, **RU**B), Bochum, Germany, June 15–17, 2016, pp. 241–256. http://www.math.bas.bg/~epopova/papers/16-EPopova-REC16.pdf

[55] Popova, E. D., Improved solution to the generalized Galilei's problem with interval loads. *Archive of Applied Mechanics*, **87** (1), 115–127 (2017). https://doi.org/10.1007/s00419-016-1180-2

[56] Popova, E. D., Interval algebraic approach to equilibrium equations in mechanics. In: Georgiev, K., Todorov, M. and Georgiev, I. (editors), *Advanced Computing in Industrial Mathematics*, Revised Selected Papers of the 10th Annual Meeting of the Bulgarian Section of SIAM, Sofia, Bulgaria, December 21–22, 2015. Book series: *Studies in Computational Intelligence (SCI)*, Vol. 681, pp. 161–173. Springer, Cham, Switzerland, 2017. https://doi.org/10.1007/978-3-319-49544-6_14

[57] Chakraverty, S., Hladík, M. and Behera, D., Formal solution of an interval system of linear equations with an application in static responses of structures with interval forces. *Applied Mathematical Modelling*, **50**, 105–117 (2017). https://doi.org/10.1016/j.apm.2017.05.010

[58] Muscolino, G., Sofi, A. and Giunta, F., Dynamics of structures with uncertain-but-bounded parameters via pseudo-static sensitivity analysis. *Mechanical Systems and Signal Processing*, **111**, 1–22 (2018). https://doi.org/10.1016/j.ymssp.2018.02.023

[59] Su, J., Zhu, Y., Wang, J., Li, A. and Yang, G., An improved interval finite element method based on the element-by-element technique for large truss system and plane problems. *Advances in Mechanical Engineering*, **10** (4), 1–10 (2018). https://doi.org/10.1177/1687814018769159

[60] Popova, E. D. and Elishakoff, I., Novel application of interval analysis in strength of materials (extended abstract). Paper presented to 13th Annual Meeting of the Bulgarian Section of SIAM, Sofia, Bulgaria, December 18–20, 2018. Hosted by the Institute of Mechanics, Bulgarian Academy of Sciences. In: *BGSIAM'18, Extended Abstracts*. Fastumprint, Sofia, Bulgaria, 2018, pp. 93–94. http://www.math.bas.bg/IMIdocs/BGSIAM/docs/bgsiam_2018_abstracts.pdf#page=103

[61] Popova, E. D., Equilibrium equations in interval models of structures. *International Journal of Reliability and Safety*, **12** (1/2), 218–235 (2018). https://doi.org/10.1504/IJRS.2018.092525; web page of the related PDF file at Semantic Scholar: https://pdfs.semanticscholar.org/db20/2872da15696cf3b493f6da827e7ba85b9e8a.pdf

[62] Popova, E. D., Algebraic solution to interval equilibrium equations of truss structures. *Applied Mathematical Modelling*, **65**, 489–506 (2019). https://doi.org/10.1016/j.apm.2018.08.021

[63] Sofi, A., Romeo, E., Barrera, O. and Cocks, A., An interval finite element method for the analysis of structures with spatially varying uncertainties. *Advances in Engineering Software*, **128**, 1–19 (2019). https://doi.org/10.1016/j.advengsoft.2018.11.001

[64] Muscolino, G. and Santoro, R., Dynamics of multiple cracked prismatic beams with uncertain-but-bounded depths under deterministic and stochastic loads. *Journal of Sound and Vibration*, **443**, 717–731 (2019). https://doi.org/10.1016/j.jsv.2018.11.029

[65] Sofi, A., Muscolino, G. and Giunta, F., Fatigue analysis of structures with interval axial stiffness subjected to stationary stochastic excitations. *Meccanica*, **54** (9) (Special Issue: *Stochastics and Probability in Engineering Mechanics*), 1471–1487 (2019). https://doi.org/10. 1007/s11012-019-01022-2

[66] Faes, M. and Moens, D., Multivariate dependent interval finite element analysis via convex hull pair constructions and the Extended Transformation Method. *Computer Methods in Applied Mechanics and Engineering*, **347**, 85–102 (2019). https://doi.org/10.1016/j.cma.2018.12.021

[67] Faes, M. and Moens, D., Recent trends in the modeling and quantification of non-probabilistic uncertainty. *Archives of Computational Methods in Engineering*, **27** (3), 633–671 (2020). https://doi.org/10.1007/s11831-019-09327-x

[68] Faes, M. and Moens, D., On auto- and cross-interdependence in interval field finite element analysis. *International Journal for Numerical Methods in Engineering*, **121** (9), 2033–2050 (2020). https://doi.org/10.1002/nme.6297

[69] Behera, D. and Chakraverty, S., Solving the nondeterministic static governing equations of structures subjected to various forces under fuzzy and interval uncertainty. *International Journal of Approximate Reasoning*, **116**, 43–61 (2020). https://doi.org/10.1016/j.ijar.2019.10.011

[70] Popova, E. D. and Elishakoff, I., Novel interval model applied to derived variables in static and structural problems. *Archive of Applied Mechanics*, **90** (4), 869–881 (2020). https://doi.org/10.1007/s00419-019-01644-8

[71] Muhanna, R. L. and Shahi, S., Uncertainty in boundary conditions—an interval finite element approach. In: Ceberio, M. and Kreinovich, V. (editors), *Decision Making under Constraints*. Springer, Cham, Switzerland, 2020, pp. 157–167. https://doi.org/10.1007/978-3-030-40814-5_20

[72] Santoro, R., Failla, G. and Muscolino, G., Interval static analysis of multi-cracked beams with uncertain size and position of cracks. *Applied Mathematical Modelling*, **86**, 92–114 (2020). https://doi.org/10.1016/j.apm.2020.03.049

[73] Rao, S. S. and Alazwari, M. A., Failure modeling and analysis of composite laminates: interval-based approaches. *Journal of Reinforced Plastics and Composites*, **39** (21–22), 817–836 (2020). https://doi.org/10.1177/0731684420932645

[74] Ni, B. Y. and Jiang, C., Interval field model and interval finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, **360** (1 March 2020), article 112713, 40 pages (2020; in press, published online on 20 November 2019). https://doi.org/10.1016j.cma.2019.112713

[75] Dinh-Cong, D., Van Hoa, N. and Nguyen-Thoi, T., An effective optimization-based parameterized interval analysis approach for static structural response with multiple uncertain parameters. *Engineering with Computers*, **36** (4), 1889–1902 (2020). https://doi.org/10.1007/s00366-019-00803-3

[76] Popova, E. D., New parameterized solution with application to bounding secondary variables in FE models of structures. *Applied Mathematics and Computation*, **378** (1 August 2020), article 125205, 14 pages (2020; in press). https://doi.org/10.1016/j.amc.2020.125205. See also arXiv: 1812.07300v2. Cornell University Library, Ithaca, New York, 2020, version 2 (preprint): https://arxiv.org/abs/1812.07300

[77] Skalna, I., *Parametric Interval Algebraic Systems*. Book series: *Studies in Computational Intelligence*, Vol. 766. Springer, Cham, Switzerland, 2018. https://doi.org/10.1007/978-3-319-75187-0

[78] Kolev, L., P-solutions of linear and nonlinear interval parametric systems and applications. Presented to the *8th International Workshop on Reliable Engineering Computing (REC), Computing with Confidence*, REC 2018, Institute for Risk and Uncertainty, University of Liverpool, Liverpool, U.K., July 16–18, 2018, 28 pages. http://rec2018.uk/papers/REC2018-21_L.Kolev__P-solutions%20of%20linear%20and%20non-linear%20interval%20parametric%20systems%20and%20applications.pdf

[79] Skalna, I. and Hladík, M., Direct and iterative methods for interval parametric algebraic systems producing parametric solutions. *Numerical Linear Algebra with Applications*, **26** (3), paper e2229, 24 pages (May 2019). https://doi.org/10.1002/nla.2229

[80] Popova, E. D., Proving endpoint dependence in solving interval parametric linear systems. *Numerical Algorithms*, 20 pages (2020; in press, published online on 12 May 2020). https://doi.org/10.1007/s11075-020-00936-3; free read-only access is provided here; related hyperlink: https://rdcu.be/b38aM.

[81] Keiper, J. B., Interval arithmetic in Mathematica. *Interval Computations*, **3**, 76–87 (1993). https://interval.louisiana.edu/reliable-computing-journal/1993/interval-computations-1993-3-pp-076-087.pdf

[82] Popova, E. D. and Ullrich, Ch. P., Directed interval arithmetic in *Mathematica*: implementation and applications. Technical Report 96-3, Universitätsrechenzentrum und Institut für Informatik (URZ+IFI), University of Basel, Basel, Switzerland, 1996, i+43 pages. Web page of the related PDF file at the Bulgarian Academy of Sciences, Sofia, Bulgaria: http://www.math.bas.bg/~epopova/papers/tr96-3.pdf

[83] Popova, E. D., Parametric interval linear solver. *Numerical Algorithms*, **37** (1–4), 345–356 (2004). https://doi.org/10.1023/B:NUMA.0000049480.57066.fa

[84] Sainz, M. A., Armengol, J., Calm, R., Herrero, P., Jorba, L. and Vehi, J., *Modal Interval Analysis: New Tools for Numerical Information* (Series: *Lecture Notes in Mathematics*, Vol. 2091). Springer, Cham, Switzerland, 2014. https://doi.org/10.1007/978-3-319-01721-1; see also the web page at ResearchGate with a link to the PDF file (preprint): https://www.researchgate.net/publication/264950076

[85] Grandón, C. and Neveu, B., A specific quantifier elimination for inner box test in distance constraints with uncertainties. Research Report RR-5883, INRIA (Institut National de Recherche en Informatique et en Automatique), Sophia Antipolis, France, 2006, 23 pages. Web page at HAL (archives-ouverts.fr): https://hal.inria.fr/inria-00071385/document

[86] Grandón, C. and Goldsztejn, A., Quantifier elimination versus generalized interval evaluation — a comparison on a specific class of quantified constraints. In: Bouchon–Meunier, B., Marsala, Ch., Rifqi, M. and Yager, R. R. (editors), *Uncertainty and Intelligent Information Systems*. World Scientific, Singapore, 2008, pp. 501–514. https://doi.org/10.1142/9789812792358_0036; web page of the related PDF file at Semantic Scholar (preprint): https://pdfs.semanticscholar.org/c60b/2a4a5d9692ad8cdea4f156c3dc1d7767d488.pdf

[87] Goldsztejn, A., Modal intervals revisited, Part 1: A generalized interval natural extension. *Reliable Computing*, **16**, 130–183 (2012). https://interval.louisiana.edu/reliable-computing-journal/volume-16/reliable-computing-16-pp-130-183.pdf

[88] Khanh, T. V. and Ogawa, M., SMT for polynomial constraints on real numbers. *Electronic Notes in Theoretical Computer Science*, **289**, 27–40 (2012). https://doi.org/10.1016/j.entcs.2012.11.004

[89] Popova, E. D., Explicit characterization of a class of parametric solution sets. *Comptes rendus de l'Académie bulgare des Sciences*, **62** (10), 1207–1216 (2009). Web page of the PDF file of a preprint of this paper (Preprint No. 1/2007, Institute of Mathematics and Informatics) at the Bulgarian Academy of Sciences, Sofia, Bulgaria, 2007: http://www.math.bas.bg/library/images/prepr/1-2007-preprint_ep

[90] Popova, E. D. and Krämer, W., Characterization of *AE* solution sets to a class of parametric linear systems. *Comptes rendus de l'Académie bulgare des Sciences*, **64** (3), 325–332 (2011). https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.449.2174&rep=rep1&type=pdf

[91] Kolev, L. V., Automatic computation of a linear interval enclosure. *Reliable Computing*, **7** (1), 17–28 (2001). https://doi.org/10.1023/A:1011470916953

[92] Krawczyk, R. and Neumaier, A., Interval slopes for rational functions and associated centered forms. *SIAM Journal on Numerical Analysis*, **22** (3), 604–616 (1985). https://www.jstor.org/stable/2157086

[93] Zuhe, S. and Wolfe, M. A., On interval enclosures using slope arithmetic. *Applied Mathematics and Computation*, **39** (1), 89–105 (1990). https://doi.org/10.1016/0096-3003(90)90124-L

[94] Kolev, L. V., Use of interval slopes for the irrational part of factorable functions. *Reliable Computing*, **3**, 83–93 (1997). https://doi.org/10.1023/A:1009902813842

[95] Kolev, L. V., An efficient interval method for global analysis of non-linear resistive circuits. *International Journal of Circuit Theory and Applications*, **26** (1), 81–92 (1998). https://doi.org/10.1002/(SICI)1097-007X(199801/02)26:1<81::AID-CTA993>3.0.CO;2-P

[96] Neumaier, A., Improving interval enclosures. Manuscript (2015 version). Fakultät für Mathematik, Universität Wien, Wien, Austria, 2015, 26 pages. https://www.mat.univie.ac.at/~neum/ms/encl.pdf

[97] Kolev, L. V., Parameterized solution of linear interval parametric systems. *Applied Mathematics and Computation*, **246**, 229–246 (2014). https://doi.org/10.1016/j.amc.2014.08.037

[98] Kolev, L., A class of iterative methods for determining *p*-solutions of linear interval parametric systems. *Reliable Computing*, **22**, 26–46 (2016). https://interval.louisiana.edu/reliable-computing-journal/volume-22/reliable-computing-22-pp-026-046.pdf

[99] Kolev, L., Parameterized solution of nonlinear interval parametric systems (Preprint_a), 2018, 17 pages. Web page at ResearchGate with a link to the related PDF file: https://www.researchgate.net/ publication/324546321

[100] Kolev, L., P-solutions for a class of structured interval parametric systems, 2018, 14 pages. Web page at ResearchGate with a link to the related DOC file: https://doi.org/10.13140/RG.2.2.15508.76167

[101] Skalna, I. and Hladík, M., A new method for computing a *p*-solution to parametric interval linear systems with affine-linear and nonlinear dependencies. *BIT Numerical Mathematics*, **57** (4), 1109–1136 (2017). https://doi.org/10.1007/s10543-017-0679-4

[102] Popova, E. D., On a class of parameterized solutions to interval parametric linear systems. *Comptes rendus de l'Académie bulgare des Sciences*, **73** (5), 599–611 (2020). https://doi.org/10.7546/CRABS.2020.05.02. See also arXiv: 1906.00613. Cornell University Library, Ithaca, New York, 2019 (preprint): https://arxiv.org/abs/1906.00613

[103] Collins, G. E., Application of quantifier elimination to Solotareff's approximation problem. In: Jeltsch, R. and Mansour, M. (editors), *Stability Theory (Hurwitz Centenary Conference, Centro Stefano Franscini, Ascona, 1995)*, ISNM (International Series of Numerical Mathematics), Vol. 121. Birkhäuser, Basel, Switzerland, 1996, pp. 181–190. https://doi.org/10.1007/978-3-0348-9208-7_19

[104] Vajda, R., Effective real quantifier elimination. In: Karsai, J. and Vajda, R. (editors), *Interesting Mathematical Problems in Sciences and Everyday Life - 2011,* pp. 1–18. University of Szeged, Szeged, Hungary. Web page at ResearchGate with a link to the related PDF file: https://www.researchgate.net/publication/267372344; see also the related web page at the University of Szeged: http://www.model.u-szeged.hu/etc/edoc/imp/RVajda/RVajda.pdf

[105] Hetényi, M., *Beams on Elastic Foundation, Theory with Applications in the Fields of Civil and Mechanical Engineering*. The University of Michigan Press, Ann Arbor, Michigan, 1946. Web page at Internet Archive with a link to the related PDF file: https://archive.org/details/in.ernet.dli.2015.73919

[106] Deif, A. S., *Advanced Matrix Theory for Scientists and Engineers*, 2nd edition. Abacus Press (an imprint of Gordon and Breach Science Publishers), Montreux, Switzerland, 1991. Web page at ResearchGate with a link to the related PDF file: https://www.researchgate.net/publication/265811236